

Impact of the Evolution of a Curved Charge Distribution on Electric Field

Haiduke Sarafian

The Pennsylvania State University, State College, USA

Email: has2@psu.edu

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Abstract

Theoretically, it is plausible to assume for a chosen charge distribution the electric field can be calculated. However, in practice depending on the geometry of the distribution one faces mathematical challenges. In this research-oriented project, we select a set of related familiar 2D geometric curves addressing the mathematical issues. Specifically, we consider a family of curves that evolved via step-by-step “evolution”. The evolution begins from a segment of a circular arc to a complete circle. The electric fields are formulated, evaluated, and graphed. Accomplishing these objectives relied heavily on utilizing a Computer Algebra System (CAS), specifically *Mathematica* [1]. The CPU’s expensive runtimes are circumvented by introducing mathematical procedures.

Keywords

Charge Distribution, Electric Field, Computer Algebra System, *Mathematica*, MapleSoft

1. Introduction

At the outset, one might envision a given charge distribution creating an electrostatic potential, ϕ , and that creates the electric field, E , via the global relationship, $E = -\nabla\phi$, [2]. If the mere objective is to hunt for the electric field naturally the latter equation is less mathematically challenging that is because once the scalar potential function, ϕ , is formed evaluation of its gradient is straightforward. However, although one of our objectives is to hunt for the field, we wish not to apply the latter procedure. That is to say directly by utilizing the charge distribution to evaluate the field. We realize this potentially may pose mathematical difficulties, meaning one needs to wrestle with vectors, however, we foresee with the advances in Computer Algebra System (CAS) the mathe-

mathematical challenges might be at ease.

For the sake of demonstration in this research-oriented project, we consider a family of related continuous charge distributions. We begin with a basic building “cell” and explore the impact of its evolution while morphing to a circular completion.

With this plan, we craft our report which is composed of three sections. In addition to the Introduction in Sect. 2, we describe the details of the project showing the evolution of the distribution, and the formulation of the problem. This section shows the CAS in action, specifically *Mathematica* and on occasion Maplesoft. The output of the codes for visual understanding is accompanied by an atlas of the graphs. We conclude the article with the last section, Conclusions and Comments.

2. Formulation and Analysis

Figure 1 depicts the systematic evolution of a circular arc. It shows it begins with a 45° arc and in eight successive steps completes a full circle. Assuming each arc length sustains the same charge density, this assists in paving the road yielding the formulation of the electric field at an exterior point along the horizontal axis.

```
Table[ParametricPlot[{rCos[θ],rSin[θ]}/.r->1.,{θ,0,nπ/4},PlotRange->
{{-1.2,1.2},{-1.2,1.2}},PlotLabel->StringJoin[θ^°=","ToString[n45]],PlotStyle->Red],{n,1,8}]
```

Figure 2 shows the issue on hand. Point *p* is a typical exterior point of interest. The charged element on the rim of the circular curve contains a minuscule charge, *dq*, in the first quadrant. It generates a differential *E*_field, *dE*, with orientation as shown.

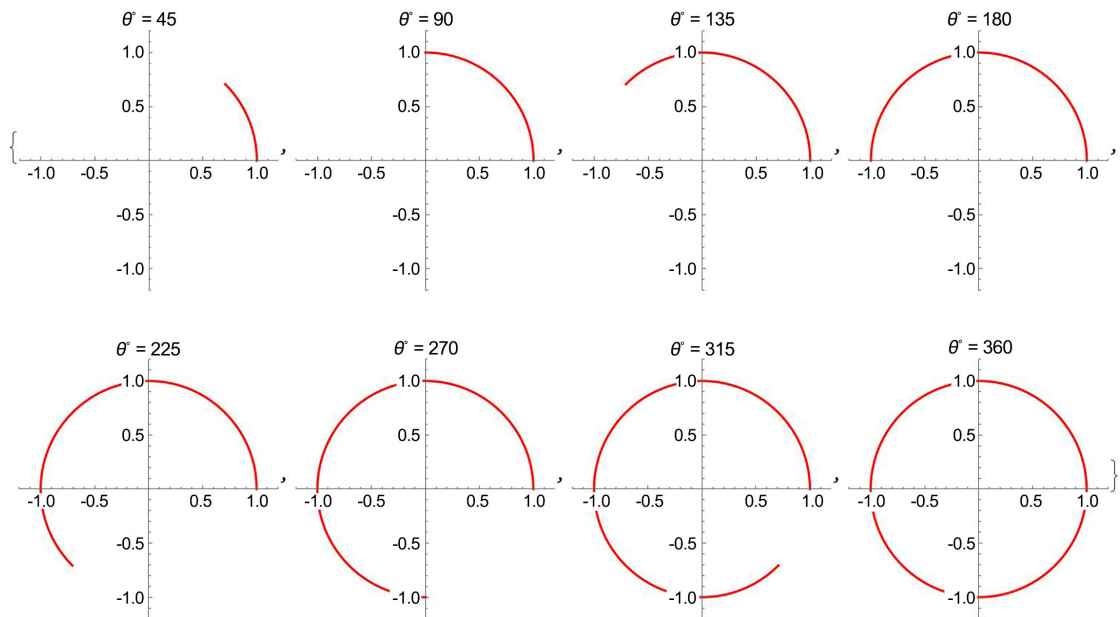


Figure 1. Evolution of a charged circular arc from 45° to completion to a complete circle.

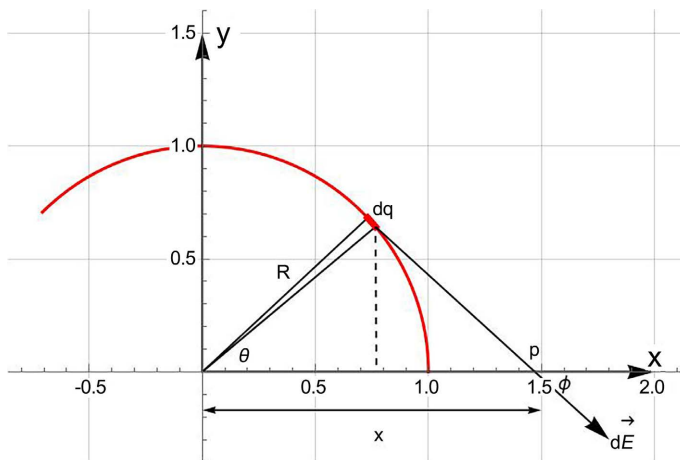


Figure 2. The semi-circular curve of radius R is in red. An element of a charge, dq , is depicted by the thick red segment. The point of interest, p , is at distance x away on the horizontal axis.

One of our objectives is to identify (evaluate) the field due to the entire charge on the circular arc. The steps needed are:

As shown in **Figure 2** assuming the differential charge dq is small as such it acts as a point charge, the overall E field comes about by adding pieces originating from the distributed charge on the rim of the curved line,

$$E = k \int \frac{dq}{dist^2} \tag{1}$$

The denominator of (1) is the distance from dq to p and the value of K in SI units is $k = \frac{1}{4\pi\epsilon_0} = 8.9 \times 10^9 \text{ N} \cdot \left(\frac{\text{m}}{\text{C}}\right)^2$. As shown the horizontal and vertical components of \mathbf{E} explicitly are,

$$E_x(x) = k\lambda R \int_0^{\theta_{up}} \frac{x - R \cos \theta}{(x^2 + R^2 - 2xR \cos \theta)^{\frac{3}{2}}} d\theta$$

$$E_y(x) = k\lambda R \int_0^{\theta_{up}} \frac{\sin \theta}{(x^2 + R^2 - 2xR \cos \theta)^{\frac{3}{2}}} d\theta \tag{2}$$

where the λ is the linear charge density, $\lambda = \frac{q}{\ell}$. As shown in **Figure 1** the lower limit of the integrations is always zero, however, the upper limits in radian, θ_{up} , correspond to the selected curve.

We begin with the case where the upper limit of (2) is a π . This corresponds to the fourth diagram on the first row of **Figure 1**. We intend to apply CAS, specifically *Mathematica* to evaluate most of the needed integrations. For an exterior point, p , i.e. for $x > r$, (1) is coded as,

```
intxπ=Integrate[(x-rCos[θ])/(x^2+r^2-2xrCos[θ])^(3/2),{θ,0,π},Assumpt
ions→x>r]
((EllipticE[(4 r x)/(r+x)^2]/(-r+x)+EllipticK[(4 r x)/(r+x)^2]/(r+x)) Sign[r+x])
/x if r+x!=0
```

We noticed 1) on an Intel COREi5, *Mathematica* V13.0 takes eight minutes to produce the shown output, and *Mathematica* V13.1 fails to produce an output. And 2) the second integration is somewhat, and its run time is short.

```
intyπ=Integrate[(Sin[θ])/(x^2+r^2-2xrCos[θ])^(3/2),{θ,0,π},Assumptions->x>r]
(r-x+Abs[r+x])/(rx(-r+x)Abs[r+x])
```

With these observations we have devised an approach circumventing these issues, we propose replacing the integrands with reasonably ordered Taylor expanded polynomials. Knowing the exact output of the upper limit of the case, π , we establish the appropriate order for the Taylor expansions. What follows is the result of our investigation.

By trial and error we realized low ordered polynomials for short distances estimate the exact electric fields however, they agreed at far distances. We optimize the order by expanding the integrand to higher orders. As such 20-ordered polynomial yields a reasonable output, needless to point out the quality of the output stays as good for longer distances. Aiming for a perfect result requires higher than 20 ordered polynomials, but this becomes CPU time expensive which we are trying to avoid.

Here are the steps leading to our procedure, Taylor expansion of the y-component of (1) for a circle of radius of one unit is,

```
Taylorx20=Normal[Series[(x-rCos[θ])/(x^2+r^2-2xrCos[θ])^(3/2),{Cos[θ],0,20}]];
integratexn=Table[Integrate[Taylorx20/.r->1,{θ,0,n π/4}],{n,1,8}];
```

The output of these codes is suppressed, as they are very long.

We also give the Maplesoft code assuming some might-be interested [3].

```
Taylor20 := series((-r*xi + x)/(-2*r*x*xi + r^2 + x^2)^(3/2), xi = 0, 20);
convert&xi;20` := convert(Taylor20, polynom);
convert&xi;&theta;20` := subs(xi = cos(theta), `convert&xi;20`);
int&theta;20` := int(`convert&xi;&theta;20`, theta = 0 .. Pi);
subs20 := subs(r = 1, `int&theta;20`);
```

Plots of the given *Mathematica* codes are shown in **Figure 3**.

```
plotxn=Table[Plot[integratexn[[n]],{x,1.01,6},AxesLabel->{"x","E_x"},GridLines->Automatic,PlotStyle->Hue[0.1 n],PlotLabel->StringJoin["integrated angle \n from 0 to=",ToString[n "π/4"]]],{n,1,8}]
```

The result shown in **Figure 4** is what intuitively one would have expected, *i.e.* the longer the arc length the stronger the associated field and vice versa. As such the brown curve with the shortest arc length is the weakest while the magenta curve corresponding to the longest arc length is the strongest. All the shown curves become indistinguishable at long distances.

For the y-axis, we repeat the mentioned procedure utilizing the second equation of (2).

```
Taylor20=Normal[Series[Sin[θ]/(x^2+r^2-2xrCos[θ])^(3/2),{Cos[θ],0,20}]];
integrateyn=Table[Integrate[Taylor20/.r->1,{θ,0,n π/4}],{n,1,8}];
```

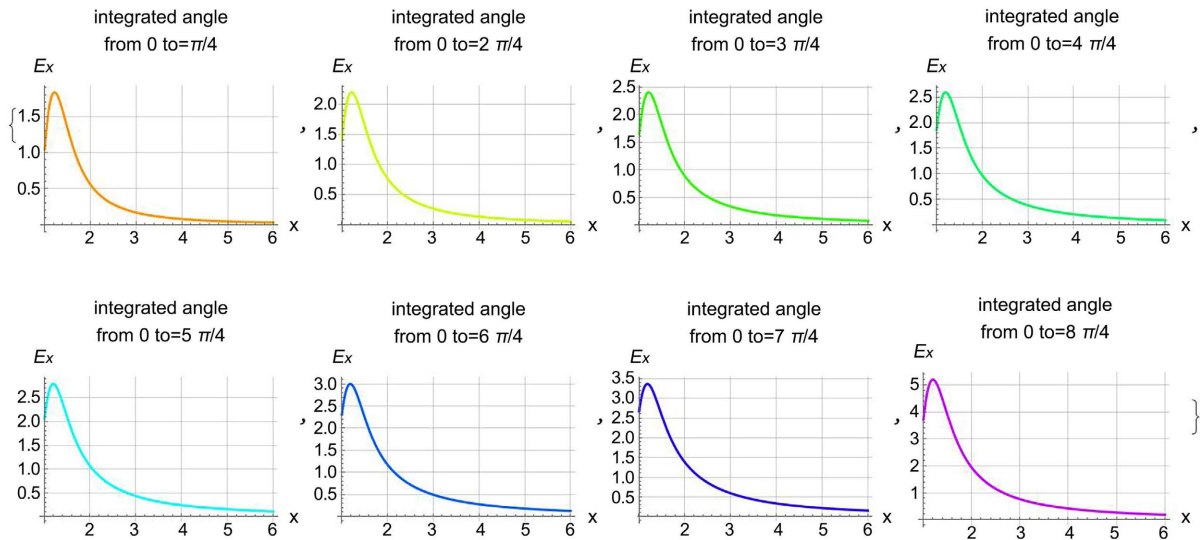


Figure 3. The Magnitudes of electric fields along the x-axis applying (2) with mentioned 20 terms Taylor expansion are depicted. The plot labels are descriptive.

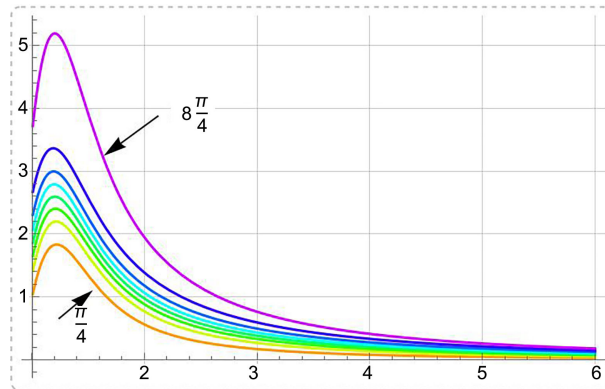


Figure 4. Collective display of all eight plates of **Figure 3**.

Here again because of the lengthiness of the outputs they are suppressed. The unitless E_y vs. the unitless x-axis are shown in **Figure 5**.

```

tablen1=Table[{x,180./π ArcTan[integrateyn[[1]]/integratexn[[1]]/.r->1.}],
{x,1.01,6}]
tablen4=Table[{x,180./π ArcTan[integrateyn[[4]]/integratexn[[4]]/.r->1.}],
{x,1.01,6}]
{{1.01,66.2426},{2.01,15.7619},{3.01,9.22861},{4.01,6.48196},{5.01,4.98543}}
{{1.01,59.9414},{2.01,18.7433},{3.01,12.2783},{4.01,9.16265},{5.01,7.31438}}
listplotn1=ListPlot[tablen1,PlotLabel->"π/4",AxesLabel->{"x(m)","φ"},Plot
Style->Blue,GridLines->Automatic];
listplotn4=ListPlot[tablen4,PlotLabel->"π",AxesLabel->{"x(m)","φ"},Plot
Style->Red,GridLines->Automatic];
Show[{listplotn1,listplotn4},PlotLabel->"Blue="π/4"\n Red="π"]
    
```

Here we display two samples of the orientation angle, ϕ , for the electric fields associated with the $\pi/4$ and π arc lengths, for the x-axis, respectively shown in **Figure 2**.

In **Figure 6**, the blue dots correspond to the arc-length $\pi/4$, the first plate of **Figure 1**. The red dots are for the arc length of π and corresponds to the last plate of the first row of **Figure 1**.

Here we report the magnitude of the electric fields for mentioned cases, for the $\pi/4$ and π cases respectively.

```
electricFieldsn1=Table[{x,sqrt(integratexn[1]^2+integrateyn[1]^2)},
{x,1.01,6}
{{1.01,2.5979},{2.01,0.574783},{3.01,0.170703},{4.01,0.0798993},{5.01,0.046053
1}}
```

```
electricFieldsn4=Table[{x,sqrt(integratexn[4]^2+integrateyn[4]^2)},
{x,1.01,6}
{{1.01,3.71365},{2.01,1.01629},{3.01,0.387647},{4.01,0.2077},{5.01,0.130107}}
```

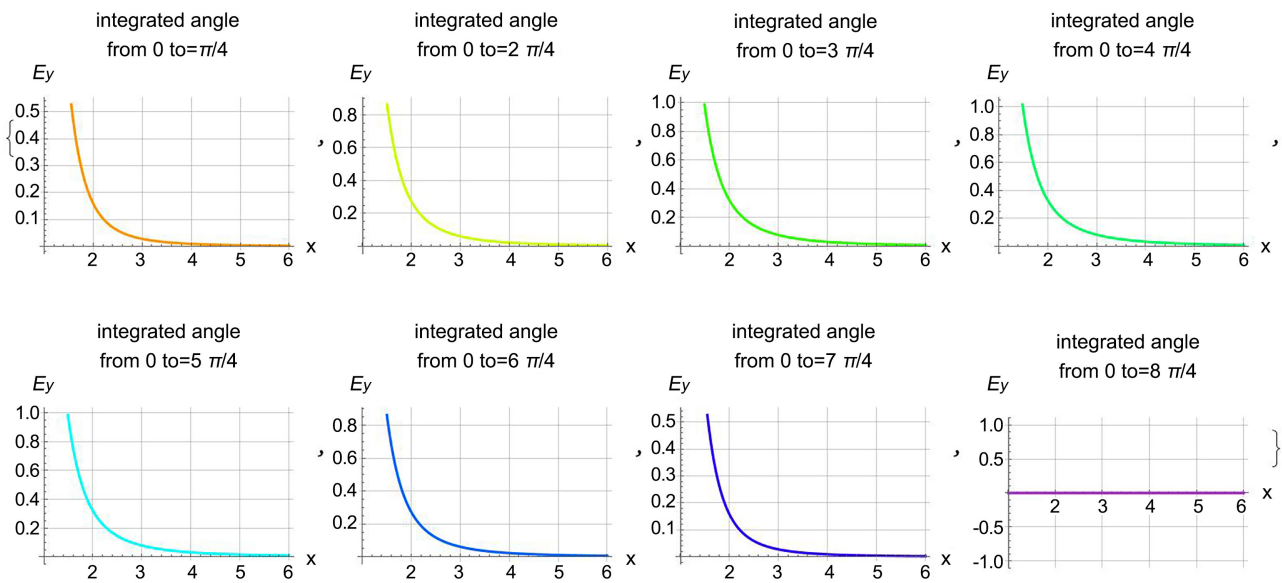


Figure 5. The Magnitudes of electric fields along the y-axis applying the second equation of (2) with 20 terms Taylor polynomial are depicted. The plot labels are descriptive.

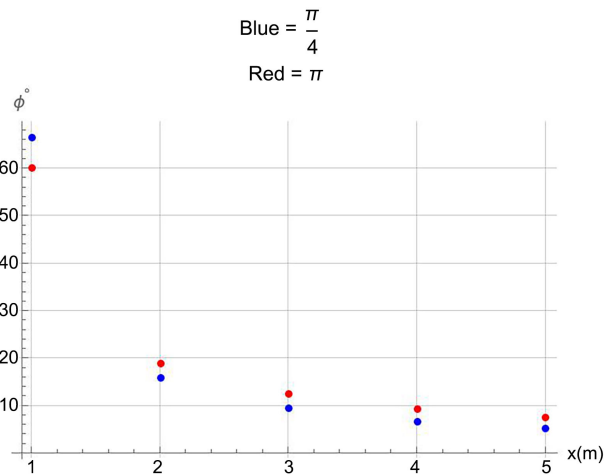


Figure 6. Display of the orientation angle of the resultant E field vs. distance.

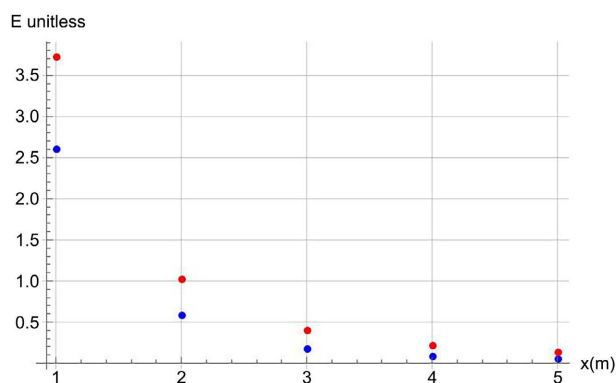


Figure 7. Display of the magnitude of E field for two cases of interest vs the distance. The blue dots correspond to $\pi/4$, while the reds with π .

```
listplotE1=ListPlot[electricFieldsn1,PlotStyle->Blue,AxesLabel->{"x(m)",
"E unitless"},GridLines->Automatic];
```

```
listplotE4=ListPlot[electricFieldsn4,PlotStyle->Red,AxesLabel->{"x(m)",
"E, unit less"},GridLines->Automatic];
```

```
Show[{listplotE1,listplotE4},PlotRange->All]
```

In **Figure 7**, as expected, the blue values are weaker than the red. This is because the former corresponds to a shorter arc length, where the reds with the longer.

3. Conclusions and Comments

In this research-oriented investigation by choosing a suitable approximation we circumvent the CPU expensive CAS-based procedure efficiently shortening the run-time of computations that we routinely face computing electrostatic-based problems. We have utilized two different CASs: *Mathematica* and MapleSoft proving identical outputs irrespective of the named software concluding the choice of the software is personal. The proposed methodology softly approximates the exact calculation and nonetheless runs the computation without hanging the computer. To make the output of the calculation as close as possible to the exact with optimizing the CPU run-time we considered 20 ordered polynomials in the Taylor approximation. This high-order replacement irrespective of the arc length of the curve of the case on hand generates perfect agreements at long distances relative to the radius of the curvature and mildly underestimates at shorter distances. The interested reader may find [4] and [5] resourceful to run the embedded codes creating the associated graphs.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this pa-

per.

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