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Electrified Bow and Arrow

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Abstract

We consider an electrostatic bow and arrow both charged positively. The bow is circular and horizontal the massive arrow is vertically aligned with the bow's symmetry axis with its head up. The arrow is released freely, the electrically charged ring repels the arrow and the gravity slows its uprise. The mass, length, and charge of the arrow as well as the size of the ring and its charge adjusted making the arrow oscillate up and down. The kinematic and dynamic quantities of the oscillations are calculated, and the relevant phase diagrams are depicted. In pursuing these goals, a Computer Algebra System (CAS) specifically *Mathematica* [1] is used. Among various scenarios, the case of a charged ring and a point charge is discussed.

Keywords

Bow and Arrow, Electrostatic Interaction, Computer Algebra System, *Mathematica*

1. Introduction

The motivation to analyze this problem stems from the fact that we have an interest in learning about the charged bodies' interaction. This objective is exercised in our work [2]. In our current investigation, two bodies, a charged arrow and a ring are considered. These shapes are considered because the output of the calculation is subject to transparent physical interpretations.

Figure 1 depicts the schematic setting of the problem on hand. The circular charged ring held horizontally acts as a bow. Its size and charge can be adjusted accordingly. The charged arrow is aligned vertically along the symmetry axis of the ring. Its length, mass, and charge also may be adjusted at wish. The outline of the calculation conducive to the electrostatic repulsive force between these two bodies calls: 1) the electric field of the ring at a representative charge element on the arrow, and 2) the superposition of this field along the length of the arrow.

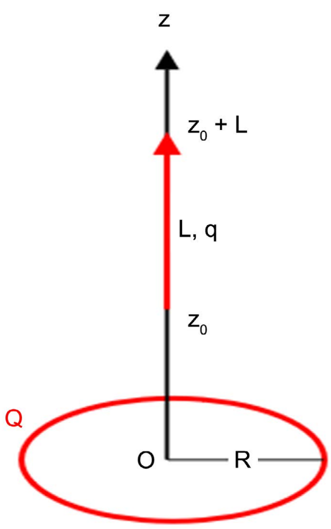

Figure 1. A charged ring of radius R is the bow and a charged arrow of length L is placed along its symmetry axis.

Hence, the net force on the arrow is the difference between the mentioned calculated repulsive electrostatic force and the gravity pull. It is obvious that by adjusting the relevant parameters, the net force may result in vertical oscillations. This is because although the gravity pulls stay the same, the electrostatic counteracting repulsive force varies depending on the proximity of the arrow to the ring. An ill-posed parametrization may either result in a departing arrow or catastrophically fall through the bow.

With this bird's eye view, the objectives of this article are crafted comprising three sections. In addition to the Introduction, Section 2 embodies the relevant formulation. This section also includes the relevant figures and helpful *Mathematica* codes. Section 3 is the conclusions and comments and the lessons learned.

2. Formulation and Analysis

The electric field of a charged ring of radius R and charge Q along its vertical symmetry through its center along the z-axis at a distance z from the ring's center is [3],

$$E(z) = kQ\frac{z}{\left(R^2+z^2\right)^{\frac{3}{2}}}, \tag{1}$$

where k is the electrostatic coupling constant, $k = \frac{1}{4\pi\epsilon_0} = 8.9\times10^9\ \mathrm{N}\cdot\left(\frac{\mathrm{m}}{\mathrm{C}}\right)^2$.

The electrostatic force of this field on the entire charged arrow of length ℓ with a charge density of $\lambda = q/\ell$ is,

$$F(z) = \frac{kQq}{\ell}\left[\frac{1}{\sqrt{R^2+z^2}} - \frac{1}{\sqrt{R^2+(z+\ell)^2}}\right], \tag{2}$$

Figure 2 shows the impact of the length of the charged arrow on the case where the arrow had no length. For this purpose and the forthcoming calculations we choose a set of practical parameters, these are stored in valuesRingℓ. Units are SI. Specifically, the ring and the arrow are 5 cm and 30 cm, respectively.

valuesRingℓ={k→9.10^9,Q→10.10^(-6),q→1.10^(-6),m→50.10^(-3),ℓ→30.10^(-2),r→5.10^(-2),g→10.};

cofℓ=kQq/mℓ/.valuesRingℓ;

Plot[Evaluate[{z/(z^2+r^2)^(3/2),1/ℓ(1/sqrt(r^2+z^2)-1/sqrt(r^2+(z+ℓ)^2))}/.valuesRingℓ],{z,0,1},PlotStyle→{Blue,Red},PlotRange→All,AxesLabel→{"z(distance)","~Force"},GridLines→Automatic]

Graphically, we have confirmed by running the ℓ to small values the red curve tends to match the blue curve as shown in Figure 3. Mathematically, in the limit of $\ell \to 0$, as expected (2) $\to$ (1).

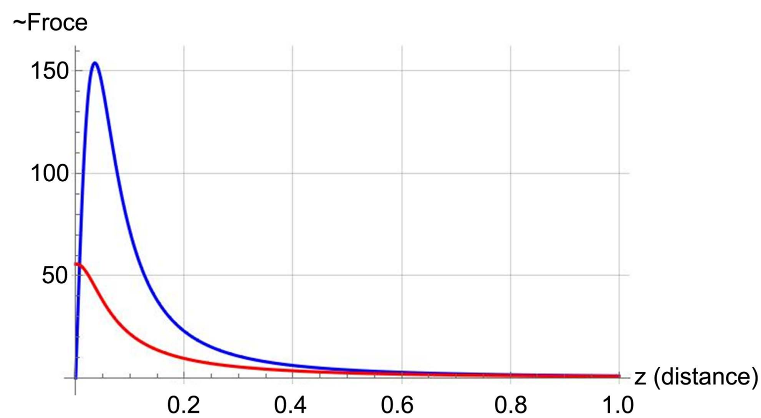

Figure 2. Generic behavior of the forces is shown vs. the distance, units are suppressed. The blue curve is the force between the ring and point charge, the red curve shows the impact of the arrow's length.

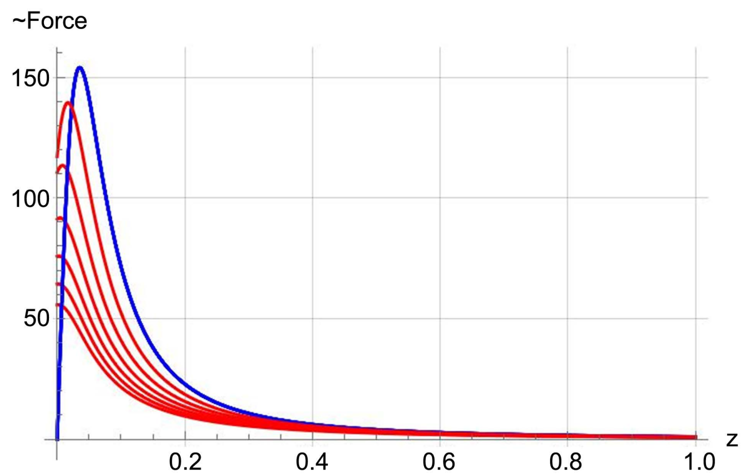

Figure 3. The blue curve is the ~Force of the charged ring at a point-like charge along the z-axis. The family of the red curves is the associated forces of the charged ring along the gradually stretched length of the arrow.

$$\frac{1}{\ell}\left[\frac{1}{\sqrt{R^2+z^2}}-\frac{1}{\sqrt{R^2+(z+\ell)^2}}\right]\xrightarrow[\ell\to 0]{}\frac{z}{\left(R^2+z^2\right)^{\frac{3}{2}}}, \tag{3}$$

In **Figure 3**, the weakest curve is associated with the longest length and the strongest with the shortest, respectively.

An arrow of mass m released freely is subject to the aforementioned electrostatic force as well the counteracting gravity pull. Applying Newton's dynamic law gives, $\boldsymbol{F}_{net}=m\ddot{\boldsymbol{z}}$. Where over double dots is the acceleration and the $\boldsymbol{F}_{net}=\boldsymbol{F}_{electic}-m\boldsymbol{g}$. Where $F_{electric}$ is subject to (2). Putting these together yields the equation of motion,

$$\ddot{z}(t)=\frac{kQq}{m\ell}\left[\frac{1}{\sqrt{R^2+z^2}}-\frac{1}{\sqrt{R^2+(z+\ell)^2}}\right]-g, \tag{4}$$

Applying *Mathematica*'s ND Solve, we solve (4) with initial conditions, $z(0)=\dot{z}(0)=0$. This is shown in **Figure 4**. The RHS of (4) is shown in **Figure 5**.

```
solzztl=NDSolve[{(z''[t]==cofℓ(1/sqrt(r^2+z[t]^2)-1/sqrt(r^2+(z[t]+ℓ)^2))-g)
/.valuesRingℓ,z[0.]==0.,z'[0.]==0.},z[t],{t,0,5}];
plotzℓ=Plot[z[t]/.solzztℓ,{t,0,2.5},AxesLabel→{"t(s)","z(m)"}(*,AxesOrigin→{0,0
}*),PlotStyle→Black,GridLines→Automatic]
```

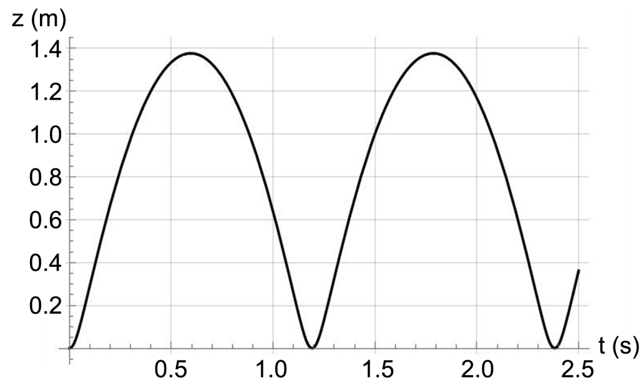

Figure 4. The oscillatory character of the bouncing arrow.

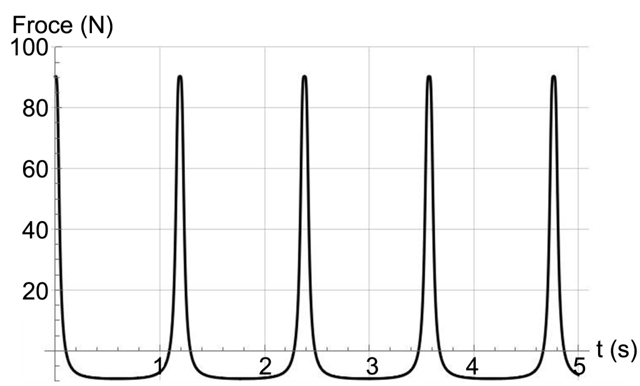

Figure 5. Display of the F_{net} vs. time.

Figure 4 shows the arrow released from rest rises to about 1.4 m, it stops for a split second and falls back to the origin. The movement is oscillatory with a 1.2 s period.

```
plotF=Plot[cofℓ(1/sqrt(r^2+z[t]^2)-1/sqrt(r^2+(z[t]+ℓ)^2))-g/.valuesRingℓ/.solzztℓ,{t,0,5},PlotStyle->Black,PlotRange->{-10,100}(*All*),AxesLabel->{"t(s)","Force(N)"},GridLines->Automatic]
```

Figure 5 shows the variation of the applied net force on the arrow. As intuitively expected at the release time the force is at its maximum, while rising its strength weakens, at some instant, it reaches the minimum and then gradually gains its strength. The process in absence of dissipation repeats itself periodically.

Solution (4) enables evaluating the relevant kinematic quantities such as the velocity and acceleration of the arrow. These are calculated and shown in Figure 6.

The shown graph is the pictorial behavior of the bouncing arrow. Based on the far left plot the other three plots are intuitively expected.

With this information on hand, we plot relevant and useful classic phase diagrams. These are shown in Figure 7. We were curious about a specific case where the length of the arrow is shrunk to a point simplifying the problem to a ring-point interaction. Analysis shows the difference between the cases is minimal. *I.e.* an arrow with a typical practical length acts almost the same as the point charge.

In Figure 7, the closed loop of the speed-position phase diagram is a typical profile of oscillations, the right plot is peculiar to the problem on hand.

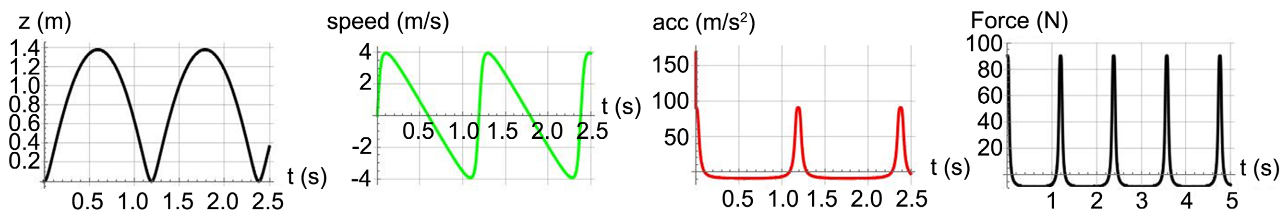

Figure 6. From left to right, respectively, the time-dependent profiles of the position, velocity, acceleration, and net force vs. time of the oscillating arrow.

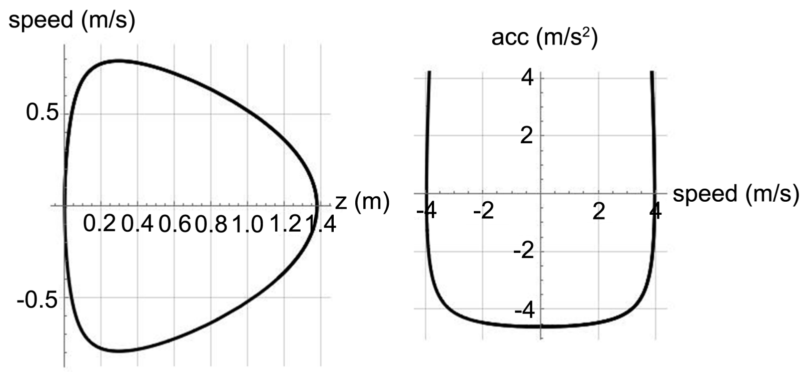

Figure 7. Phase diagrams of the oscillating arrow.

3. Conclusion and Comments

The lesson learned from investigating this research-oriented project is that we have shown for the given scenario how the electrostatic interaction between two charged bodies is formulated. Utilizing this information, we extended the static setting to a dynamic situation conducive to determining its associated classic kinematic quantities. Knowing the latter, the proposed scenario in light of the determined quantities could become practical. As such the electrostatic bow and arrow problem is envisioned, its solution and formulation add to the body of knowledge. This project can be exercised and extended by replacing the circular bow with a charged square, rectangle, and ellipse. The calculation for the first two suggestions with the aid of the embedded *Mathematic* codes may be straight forward. However, in the latter case, due to curvature dependency of the charge, distribution would be challenging. These issues partially have been addressed [4]. Concerning crafting the *Mathematica* codes, graphs, etc. the interested reader may find [5] and [6] resourceful.

Acknowledgements

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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