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Geometric Structure and Some Exact Solutions of Plateau Equation

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Authors' contributions

This work was carried out in collaboration between all authors. Author MN designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Author NA managed the analyses of the study and managed the literature searches. Both authors read and approved the final manuscript.

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ABSTRACT

In this paper, we get the set of symmetry of Plateau differential equation. Using Lie symmetry method to obtain the classical symmetry operators. Also, we get one-dimensional optimal system of the Plateau equation and reduction Lie invariants, corresponding to infinitesimal symmetries.

Keywords: Classic lie symmetry; group-invariant solution; plateau equation; optimal system.

1. INTRODUCTION

One of the most important discoveries of Sophus Lie, in differential equation is to show that, it is possible to transform non-linear conditions in a system, to linear conditions, by infinitesimal invariants, corresponding to the symmetry group generators, of the system [1]. In this article, our aim is to obtain a set of symmetries of Plateau equation [2,3]:

$$(1+u_x^2)u_{yy}-2u_xu_yu_{xy}+(1+u_y^2)u_{xx}=0 \qquad (1)$$

Which it governs, under appropriate hypotheses the plane motion of a fluid. The function u appearing in the equation is so-called kinetic potential [4] and the special case of the plateau

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equation may be used to derive a relation from the interfacial geometry of watting fluid in a capillary [5]. Minimal surfaces are defined as surfaces with zero mean curvature. The minimal surfaces PDE of the surface given by equation $u = f(x, y)$ has the Plateau equation form [6,7]. The classical Lie symmetries are obtained using the Lie symmetry method. This requires the utilization of computer softwares, because working with continuous groups has computations that follow from the algorithmic process. Having the symmetry group of a system of equations, has a lot of advantages, one of which is the classification of the solutions of the system. This classification is to consider, two solutions in one class if they can be converted to each other, by an element of the symmetry group. If we have an ordinary system, the symmetry group will help us to obtain the exact solution. If the equation is order one, it is possible to get the general solution, but it is not the case for PDE, unless the system is convertible to a linear system. Another application of the symmetry group is the probable reduction of the number of independent variables and the ideal condition is converting to ODE.

2. LIE SYMMETRY OF PLATEAU EQUATION

The method of determining the classical symmetries of a partial differential equation is standard and is described in [8,9,10]. To obtain the symmetry algebra of (1), we take an infinitesimal generator of symmetry algebra of the form:

$$v = \xi(x, y, u)\frac{\partial}{\partial x} + \eta(x, y, u)\frac{\partial}{\partial y} + \phi(x, y, u)\frac{\partial}{\partial u} \tag{2}$$

Using the invariant condition, i.e., applying the fourth prolongation $v^{(2)}$ to (1), the following *system of 15 determining equation yields:*

$$\begin{array}{lll} \xi_x - \phi_u = 0, & \phi_{xx} - \eta_{yy} = 0 & \phi_{uu} - \eta_{uy} + \phi_{xx} = 0, \\ \eta_u + \phi_y = 0, & \xi_{uu} + \xi_{yy} = 0, & \eta_{yy} - 2\phi_{uy} + \eta_{xx} = 0, \\ \xi_u + \phi_x = 0, & \eta_{uu} + \eta_{xx} = 0, & \xi_{xx} - 2\eta_{xy} + \xi_{uu} = 0, \\ \eta_x + \xi_y = 0, & \xi_{xx} - 2\phi_{ux} + \xi_{yy} = 0, & \phi_{uu} - 2\xi_{ux} + \phi_{yy} = 0, \\ \eta_y - \phi_u = 0, & \eta_{ux} + \phi_{yx} + \xi_{uy} = 0, & \eta_{uu} - 2\xi_{yx} + \eta_{yy} = 0. \end{array} \tag{3}$$

By solving the above system we will have the following theorem.

Theorem 2.1: The Lie group of point symmetries of the Plateau equation, has a Lie algebra generator in the form of the vector field v, with the following functional coefficients.

$$\begin{aligned} \xi(x, y, u) &= c_4 x - c_1 y + c_5 u + c_6, \\ \eta(x, y, u) &= c_1 x + c_4 y + c_2 u + c_3, \\ \phi(x, y, u) &= -c_5 x - c_2 y + c_4 u + c_7. \end{aligned}$$

Where c_i, $i = 1, \cdots, 7$ are arbitrary constants.

Theorem 2.2: The infinitesimal generators from the Lie one-parameter group of the symmetries of the Plateau equation are as follows:

$$v_1 = \frac{\partial}{\partial x}, \qquad v_2 = \frac{\partial}{\partial y}, \qquad v_3 = \frac{\partial}{\partial u}, \qquad v_4 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},$$
$$v_5 = u\frac{\partial}{\partial x} - x\frac{\partial}{u}, \qquad v_6 = u\frac{\partial}{\partial y} - y\frac{\partial}{\partial u}, \qquad v_7 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u}.$$

These vector fields produce a Lie algebra space $\mathcal{G}$ with the following commutator Table:

Table 1. Commutation relations satisfied by infinitesimal generators

$[\,.\,,\,.\,]$	$\mathbf{v_1}$	$\mathbf{v_2}$	$\mathbf{v_3}$	$\mathbf{v_4}$	$\mathbf{v_5}$	$\mathbf{v_6}$	$\mathbf{v_7}$
v_1	0	0	0	v_2	$-v_3$	0	v_1
v_2	0	0	0	$-v_1$	0	$-v_3$	v_2
v_3	0	0	0	0	v_1	v_2	v_3
v_4	$-v_2$	v_1	0	0	$-v_6$	v_5	0
v_5	v_3	0	$-v_1$	v_6	0	$-v_4$	0
v_6	0	v_3	$-v_2$	$-v_5$	v_4	0	0
v_7	$-v_1$	$-v_2$	$-v_3$	0	0	0	0

3. GROUP INVARIANT SOLUTIONS OF PLATEAU EQUATION

To obtain the group of transformations which are generated by infinitesimal generators v_i, $i = 1, \cdots, 7$, we should solve the first order system involving first order equations in correspondence to each of the generators simultaneously.

By solving this system, the one parameter group of $g_k(s): M \to M$ generated by v_i, $i = 1, \cdots, 7$ involved in theorem (2.2) is obtained in the following way;

$$\begin{aligned}
g_1 &: (x,y,u) \mapsto (x+s, y, u),\\
g_2 &: (x,y,u) \mapsto (x, y+s, u),\\
g_3 &: (x,y,u) \mapsto (x, y, u+s),\\
g_4 &: (x,y,u) \mapsto (y\sin s + x\cos s\,, y\cos s - x\sin s, u\,),\\
g_5 &: (x,y,u) \mapsto (u\sin s - x\cos s\,, y, u\cos s + x\sin s\,),\\
g_6 &: (x,y,u) \mapsto (x, y\cos s - u\sin s\,, u\cos s + y\sin s\,),\\
g_7 &: (x,y,u) \mapsto (x\,e^s, y\,e^s, u\,e^s).
\end{aligned}$$

Therefore, we will have the following theorem:

Theorem 3.1: If $u = f(x,y)$ is one solution of Plateau equation, then the following functions that have been produced through acting $g_k(s)$ on $u = f(x,y)$ will also be the solution of Plateau equation.

$$\begin{aligned}
g_1(s)\,.\,f(x,y) &= f(x+s, y),\\
g_2(s)\,.\,f(x,y) &= f(x, y+s),\\
g_3(s)\,.\,f(x,y) &= f(x,y) - s,\\
g_4(s)\,.\,f(x,y) &= f(y\sin s + x\cos s, y\cos s - x\sin s),\\
g_7(s)\,.\,f(x,y) &= f(x\,e^s, y\,e^s)e^{-s}.
\end{aligned}$$

4. OPTIMAL SYSTEM OF ONE-DIMENSIONAL SUBALGEBRAS OF PLATEAU EQUATION

Now we want to obtain one-dimensional optimal system of the Plateau equation using its symmetry group. The optimal system is in fact a standard method for the classification of one-dimensional sub-algebras in which each class involves conjugate equivalent members [11]. Also, they involve the group adjoint representation which establishes an equivalent relation among all conjugate sub-algebra elements. In fact, the classification problem for one-dimensional sub-algebra is the same as the problem of the classification of the representation of its adjoint orbits. In this way, the optimal system is constructed. The set of invariant solutions corresponding to a one-dimensional sub-algebra is a list of minimal solutions, where all the other invariant solutions can be obtained by transformations [12]. To calculate the adjoint representation, we consider the following Lie series:

$$Ad(\exp(s\, v_i)v_j) = v_j - s\, ad_{v_i}v_j + \frac{s^2}{2}\, ad_{v_i}^2 v_j - \cdots,$$

for the vector fields v_i , v_j in which $ad_{v_i}v_j = [v_i, v_j]$ is the Lie algebra communicator, s is the group parameter, $i\,, j = 1, \cdots, 7$ ([8], pp 199). Now, we consider an optional member from $\mathcal{G}$ of the form $v = a_1v_1 + \cdots + a_7v_7,$ and for simplicity we write $a = (a_1, \cdots, a_7) \in \mathbb{R}^7$; therefor, the adjoint action can be considered as a type of linear transformation group of vectors, so we have the following theorem:

Theorem 4.1: The one-dimensional optimal system of Lie algebra $\mathcal{G}$ for the Plateau equation is:

$(1){:}\, v_1 + a\, v_6\,,$ $\quad (2){:}\, v_2 + a\, v_5,$ $\quad (3){:}\, v_3 + a\, v_4,$ $\quad (4){:}\, v_4 + a\, v_7\,,$
$(5){:}\, v_5 + a\, v_7,$ $\quad (6){:}\, v_6 + a\, v_7,$ $\quad (7){:}\, a\, v_7.$

Where $a \in \mathbb{R}$ is arbitrary constant.

Proof. We define the map $F_i^s : \boldsymbol{\mathcal{G}} \to \boldsymbol{\mathcal{G}}$ by $v \mapsto Ad\,(\exp(s\, v_i)v)$ as a linear map, for $i = 1, \cdots, 7$. So the matrices M_i^s corresponding to each of the F_i^s, $i = 1, \ldots, 7$, in relation to the basis $\{v_1, \cdots, v_7\}$ will be as follows:

$$\begin{aligned}
M_1^s &= I - s(E_{42} + E_{71}),\\
M_2^s &= I + s(E_{41} + E_{63} - E_{72}),\\
M_3^s &= I - s(E_{51} + E_{62} + E_{73}),\\
M_4^s &= \cos s\,(E_{11} + E_{22} + E_{55} + E_{66}) + \sin s\,(E_{12} - E_{21} + E_{56} - E_{65}) + E_{33} + E_{44} + E_{77},\\
M_5^s &= \cos s\,(E_{11} + E_{33} + E_{44} + E_{66}) + \sin s\,(-E_{13} + E_{31} - E_{46} + E_{64}) + E_{22} + E_{55} + E_{77},\\
M_6^s &= \cos s\,(E_{22} + E_{33} + E_{44} + E_{55}) \sin s\,(-E_{23} + E_{32} + E_{45} - E_{54}) + E_{11} + E_{66} + E_{77},\\
M_7^s &= e^s(E_{11} + E_{22} + E_{33}) + E_{44} + E_{55} + E_{66} + E_{77}.
\end{aligned}$$

In it, E_{ij}s are

7×7-elementary matrixes, for $i, j = 1, \ldots, 7$; on the condition that the (i, j)-entry of E_{ij} is 1, and others are zero. Suppose $v = a_1v_1 + \cdots + a_7v_7$, in this case, we will have the map combinations as follows:

$$F_7^{s7} \circ F_6^{s6} \circ F_5^{s5} \circ F_4^{s4} \circ F_3^{s3} \circ F_2^{s2} \circ F_1^{s1} : v \mapsto [\cos s_4 \cos s_5\, e^{s_7}\, a_1 + (\sin s_4 \cos s_6 - \cos s_4 \sin s_5 \sin s_6) e^{s_7}\, a_2 + (-\sin s_4 \sin s_6 - \cos s_4 \sin s_5 \cos s_6)\, e^{s_7}\, a_3]\, v_1 + \cdots + [((s_2 \sin s_4 - s_1 \cos s_4) \cos s_5 - s_3 \sin s_5)\, e^{s_7}\, a_1 + (-s_1 \sin s_4 - s_2 \cos s_4) \cos s_6 + (-(s_2 \sin s_4 - s_1 \cos s_4) \sin s_5 - s_3 \cos s_5) \sin s_6\, e^{s_7}\, a_2 + (-(-s_1 \sin s_4 - s_2 \cos s_4) \sin s_6 + (-(s_2 \sin s_4 - s_1 \cos s_4) \sin s_5 - s_3 \cos s_5) \cos s_6)\, e^{s_7}\, a_3 + a_7]\, v_7.$$

We can simplify v as follows:

If $a_1 \neq 0$ then we can vanish the coefficient of $v_7,\ v_4,\ v_5,\ v_2$ and v_3 using $F_1^{s1},\ F_2^{s2},\ F_3^{s3},\ F_4^{s4}$ and F_5^{s5} by substitution $s_1 = \frac{a_7}{a_1},\ s_2 = -\frac{a_4}{a_1},\ s_3 = \frac{a_5}{a_1},\ s_4 = \arctan\frac{a_2}{a_1}$, and $s_5 = -\arctan\frac{a_3}{a_1}$. And if necessary, we can suppose $a_1 = 1$ through the scaling of v. In this case, v is reduced to form (1). If $a_1 = 0$ and $a_2 \neq 0$, then we can vanish the coefficient of $v_4,\ v_7,\ v_6$, and v_3 using $F_1^{s1},\ F_2^{s2},\ F_3^{s3}$, and F_6^{s6} by substitution $s_1 = \frac{a_4}{a_2},\ s_2 = \frac{a_7}{a_2},\ s_3 = \frac{a_6}{a_2}$, and $s_6 = -\arctan\frac{a_3}{a_2}$. And if necessary, we can suppose $a_2 = 1$ through the scaling of v. In this case, v is reduced to form (2). If $a_1 = 0$ and $a_2 = 0$ and $a_3 \neq 0$, then we can vanish the coefficient of $v_5,\ v_6$, and v_7 using $F_1^{s1},\ F_2^{s2}$, and F_3^{s3} by substitution $s_1 = -\frac{a_5}{a_3},\ s_2 = -\frac{a_6}{a_3}$, and $s_3 = \frac{a_7}{a_3}$. And if necessary, we can suppose $a_3 = 1$ through the scaling of v. In this case, v is reduced to form (3). And if $a_1 = 0,\ a_2 = 0, a_3 = 0$, and $a_4 \neq 0$, then we can vanish the coefficient of v_5, and v_6 using F_5^{s5}, and F_6^{s6} by substitution $s_5 = -\arctan\frac{a_6}{a_4}$, and $s_6 = -\arctan\frac{a_5}{a_6}$. And if necessary, we can suppose $a_4 = 1$ through the scaling of v. In this case, v is reduced to form (4). And if $a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_4 = 0$, and $a_5 \neq 0$, then we can vanish the coefficient of v_6, using F_4^{s4}, by substitution $s_4 = \arctan\frac{a_6}{a_5}$. And if necessary, we can suppose $a_5 = 1$ through the scaling of v. In this case, v is reduced to form (5). And if $a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_4 = 0,\ a_5 = 0$, and $a_6 \neq 0$, if necessary, we can suppose $a_6 = 1$ through the scaling of v. In this case, v is reduced to form (6). And if $a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_4 = 0,\ a_5 = 0$, and $a_6 = 0$, then v is reduced to form (7).

5. SIMILARITY REDUCTION OF PLATEAU EQUATION

The Plateau equation has been stated with the $(x, y; u)$ coordinate, but we are looking for a new coordinate that the equation will reduce, if we write it in the new coordinate. This new coordinate is obtained through (t, v) dependent invariant corresponding to the infinitesimal symmetry generator. If we state the Plateau equation with the new coordinate, using the chain rule, a reduced equation will result. Now we calculate the invariants corresponding to the symmetry generators existing in the optimal system. The first status for the one element of the optimal system is v_7. It has the determining equation in the form $\frac{dx}{x} = \frac{dy}{y} = \frac{du}{u}$. Solving this equation will result in the two invariants of $t = \frac{y}{x},\ v = \frac{u}{x}$. Now, if we consider $u(x, y) = v(t)x$ as a function of $t = \frac{y}{x}$, we can state the derivatives of u with respect to x and y, in the form of v and t and the derivatives of v with respect to t. If we substitute it in the Plateau equation, the Plateau equation turns to an ordinary equation as: $(1 + t^2 + v^2)v_{tt} = 0$. In the Table below, the invariants corresponding to each of elements of symmetry group and optimal system have been calculated. By substituting t_i and v_i and the derivative of v_i in respect to t_i, in place of u, x, y and the derivative of u in respect to x, y, in the Plateau equation, it turns into an ODE.

Table 2. Similarity reduced equations to ODE

OP_i	t_i	v_i	u_i	ODE
v_1	y	u	$f(t)$	$v_{tt} = 0$
v_2	x	u	$f(t)$	$v_{tt} = 0$
v_4	$x^2 + y^2$	u	$f(t)$	$t\, v_{tt} + v_t + 2t\, v_t^3 = 0$
v_5	y	$u^2 + x^2$	$\sqrt{f(t) - x^2}$	$v\, v_{tt} - v_t^2 - 2v = 0$
v_6	x	$u^2 + y^2$	$\sqrt{f(t) - y^2}$	$v\, v_{tt} - v_t^2 - 2v = 0$
v_7	$y\, x^{-1}$	$u\, x^{-1}$	$x\, f(t)$	$(1 + t^2 + v^2)v_{tt} = 0$
$v_3 + v_4$	$x^2 + y^2$	$u + \arctan(x/y)$	$f(t) - \arctan(x/y)$	$(2t + 2t^2)v_{tt} + 4t^2 v_t^3 + (2t + 3)v_t = 0$

In the above Table, it has been tried to calculate the reduced ODE corresponding to each of invariants. Also, some solutions of these ODE's (after transfer them into $(x, y; u)$ coordinate) have been given as examples.

$$u(x,y) = ay + b\,,$$
$$u(x,y) = ax + b\,,$$
$$u(x,y) = ay + bx\,,$$
$$u(x,y) = -i\, x + \frac{a}{y}\,,$$
$$u(x,y) = \sqrt{\frac{y^2}{x^2} + 1}\; i\,,$$
$$u(x,y) = -\frac{a(-y + i\, x)}{x^2 + y^2}\,,$$
$$u(x,y) = a(x + i\, y)^{\left(-\frac{1}{2} + \frac{1}{2} i\right)},$$

$$u(x,y) = \frac{1}{2}\sqrt{8a^2 - 4x^2 + 2ae^{\frac{y+b}{a}} + 2a^4 e^{-\frac{y+b}{a}}}\,,$$
$$u(x,y) = \frac{1}{2}\sqrt{8a^2 - 4y^2 + 2ae^{\frac{x+b}{a}} + 2a^4 e^{-\frac{x+b}{a}}}\,,$$
$$u(x,y) = \pm \ln\left(-2a^2 + x^2 + y^2 + \sqrt{(x^2 + y^2)(x^2 + y^2 + 4a^2)}\; i\right) + b\,,$$

$$u(x,y) = \ln\left(\frac{a + 2a(x^2 + y^2) - 4 + 2\sqrt{\left(-4 + a(x^2 + y^2)\right)(1 + x^2 + y^2)}\,\sqrt{a}}{2\sqrt{a}}\right) + \frac{1}{2}\arctan\left(\frac{-8 + (a - 4)(x^2 + y^2)}{4\sqrt{\left(-4 + a(x^2 + y^2)\right)(1 + x^2 + y^2)}}\right) + b$$

6. CONCLUSION

In this paper we obtained the Lie point symmetries of the Plateau equation by using the Lie symmetry method. Also computed the one-dimensional optimal system. This led to reducing the Plateau equation to ODE's and computing the invariants of Plateau equation.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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