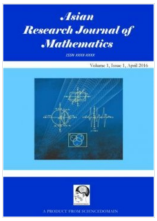

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A Study of Some Results on Countable Sets

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Authors' contributions

This work was carried out in collaboration between both authors. Author MMA designed the study, managed the section consisting of the theorems on countable sets and wrote the protocol. Author OKO managed the analyses of the study, managed the section consisting of the applications of the theorems studied and also managed the literature searches. Both authors read and approved the final manuscript.

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Abstract

In this research paper, we were able to study countable sets. To achieve this, fundamental ideas and concepts from set theory and mathematical analysis were considered. Some important theorems on countable sets were reviewed and finally, the application of the theorems studied were provided.

Keywords: Countable sets; injective functions; surjective functions; bijective functions.

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1 Introduction

The concept of a set was rather elementary one that had been used implicitly since the beginning of mathematics, dating back to the ideas of Aristotle. No one had realized that set theory had nontrivial content. Before Cantor, there were only finite sets (which are easy to understand) and "the infinite " (which was considered a topic for philosophical, rather than mathematical discussion). By Proving that there are (infinitely) many possible sizes for infinite sets, Cantor established that set theory was not trivial and it needed to be studied. Set theory has come to play the role of a foundational theory in modern mathematics, in the sense that it interprets propositions about mathematical objects for example, numbers and functions. From all the traditional areas of mathematics such as algebra, analysis and topology in a single theory and provides a standard set of axioms to prove or disprove them. The basic concepts of set theory are now used throughout mathematics.

In one of his earliest papers, proved that the set of real numbers is "more numerous" than the set of natural numbers; this showed, for the first time, that there exist infinite sets of different sizes [1]. He was also the first to appreciate the importance of one to one correspondences in set theory. He used this concepts to define finite and infinite sets, subdividing the latter into denumerable (or countably infinite) sets and uncountable sets (nondenumerable infinite sets). This notion of denumerable and nondenumerable sets led to the concept of countable and uncountable sets, a concept which is of interest to us in this paper.

The development of countability of sets was built upon the established concept of set theory. Set theory had its beginning in the 19th century transformation of mathematics, a transformation beginning in analysis. Since the creation of calculus by Newton and Leibniz, the function concept had been steadily extended from analytic expressions towards arbitrary correspondences [2]. The first major expansion had been inspired by the explorations of Euler in the 18th century and featured the infusion of infinite series methods and the analysis of physical phenomena, like the vibrating strings.

In the 19th century the stress brought on by the unbridled use of series of functions led first cauchy and then weierstress to articulate convergence and continuity.

Working out of this tradition, Georg Cantor in 1870 established a basic uniqueness theorem for trigonometric series. If such a series converges to zero everywhere, then all of its coefficients are zero [3]. To generalize, Cantor started to allow points at which convergence fails, getting to the following formulations : for a collection p of real numbers, let p' be the collection of limit points of p, and $p^{(n)}$ the result of n iteration of this operation. If a trigonometric series converges to zero everywhere except on p, where $p^{(n)}$ is empty for some n, then all of its coefficient are zero [4]. It was in 1872 that Cantor provided his formulation of the real numbers in terms of fundamental sequences of rational numbers and significantly, this was for the specific purpose of articulating his proof. With the new results of analysis to be secured by proof and proof in turn to be based on prior principles; the regress led in early 1870's to the appearance of several independent formulations of the real numbers in terms of the rational numbers. It is at first quite striking that the real numbers came to be developed so late, but this can be viewed as part of the expansion of the function concept which shifted the emphasis from the continuum taken as a whole to its extensional construal as a collection of objects [1]. In mathematics, objects have been traditionally introduced only with reluctance, but a more arithmetical rather than geometrical approach to the continuum became necessary for the articulation of proofs.

The other well–known formulation of real numbers is due to Richard Dedekind, through his cuts. Cantor and Dedekind, maintained a fruitful correspondence, especially during the 1870's in which Cantor aired many of his results and speculations [5]. The formulations of the real numbers advanced three important predispositions for set theory. The consideration of infinite collections, their construal as unitatry objects, and the encompasing of arbitrary such possibilities. Dedekind had infact made these moves in his creation of ideals, infinite collections of algebraic numbers, and there is an evident similarity between ideals and cuts in the creation of new numbers out of the old [6]. The algebraic numbers would soon be the focus of a major breakthrough by Cantor [7]. Although both cantor and Dedekind carried out an arithmetical reduction of the continuum, they each accommodated its antecedent geometric sense by asserting that each of their real numbers actually corresponds to a point on the line. Neither theft nor honest toil suffice; Cantor and Dedekind recognized the need for an axiom to this effect, a sort of church's thesis of adequacy for the new construal of the continuum as a collection of objects. Cantor recalled that around this time he was already considering infinite iterations of his p' operation using "*symbol of infinity*" [8].

$$P^{(\infty)} = \bigcap_{n}^{\infty} p^{(n)}, p^{(\infty+1)} = p^{(\infty)'}, p^{(\infty+2)}, \ldots p^{(\infty\cdot 2)}, \ldots p^{(\infty^2)}, \ldots p^{(\infty^\infty)}, \ldots p^{(\infty^{\infty^\infty})}$$

In a crucial conceptual move, he began to investigate infinite collections of real numbers and infinitary enumerations for their own sake, and this led first to a basic articulation of size for the continuum and then to a new, encompassing theory of *counting*. Set theory was born on that December 1873 day when Cantor established that *the real numbers are uncountable* [9]. In the next decades the subject was to blossom through the prodigious progress made by him in the theory of transfinite and cardinal numbers.

The uncountability of the reals was established of course, via *reductio ad absurdum* as with the irrationality of $\sqrt{2}$. Both impossibility results epitomize how a reductio can compel a larger mathematical context allowing for the deniability of hitherto implicit properties. Be that as it may, Cantor the mathematician addressed a specific problem, embedded in the mathematics of time, in his seminar entitled "on a property of totality of all real algebraic numbers". After first establishing this property, the countability of the algebraic numbers, Cantor then established : for any (countable) sequence of reals, every interval contains a real not in the sequence. Cantor appealed to the order completeness of the reals: suppose that s is a sequence of reals and I an interval. Let $a < b$ be the first two reals of s, if any, in I. Then let $a' < b'$ be the first two reals of s, if any, in the open interval (a, b); $a'' < b''$ the first two reals of s, if any, in (a', b'); and so forth. Then however long this process continues, the (non-empty) intersection of these nested intervals cannot contain any member of s.

By these means, Cantor provided a new proof of Joseph Liouville's result that there are transcendental numbers (real non-algebraic numbers) and only afterward did Cantor point out the uncountability of the reals altogether.

This presentation is suggestive of Cantor's natural caution in overstepping mathematical sense at the time [10].

Accounts of Cantor's work have mostly reversed the order for deducing the existence of transcendental numbers [11]. In textbooks the inversion may be inevitable but this has promoted the misconception that Cantor's argument are non constructive. It depends how one takes a proof, and Cantor's arguments have been implemented as algorithms to generate the successive digits of new reals [12]. Motivated by the above literature, we seek in this work to understand what countable sets are by studying the major theorems concerning countable sets.

The aim of this work is to show the applications of one of the most crucial concepts in mathematics, "countability of sets"; in order to achieve this, we studied the major theorems concerning countable sets and some applications of the theorems on sets were shown. To study this concept we shall first define some terms related to the notion.

1.1 Definition of terms

Definition 1.1. A set is a collection of well define objects, called the elements or members of the set.

For the purpose of this work; these objects are mathematical objects such as numbers or sets of numbers.

Thus, sets A, B are equal, written as $A = B$ if, $a \in A$ if and only if $a \in B$. It is convenient to define the empty set, denoted by $\emptyset$, as the set with no elements, where the set of natural numbers denoted as $\mathbb{N}$, the set of integers denoted as $\mathbb{Z}$, the set of rational numbers denoted as $\mathbb{Q}$, e.t.c are all examples of sets.

Definition 1.2. The union of sets A and B, is the set which consists of elements that are either in A or B or both. The set notation for the operation of union is $\cup$. Thus A union B is written as $A \cup B$. In set theoretical notation, $A \cup B = \{\, x : x \in A \text{ or } x \in B \text{ or } x \in \text{both } A \text{ and } B\}$

Definition 1.3. The intersection of two sets A and B; is the set which consists of elements that are in A as well as in B. The set notation for the operation of intersection is $\cap$. $A \cap B$ means; A intersection B. In set theoretical notation, the set $A \cap B = \{\, x : x \in A \text{ and } x \in B\,\}$

Definition 1.4. A set A is a subset of a set B, written as $A \subset B$ or $B \supset A$, if every element of A belongs to B.

Definition 1.5. Two sets A and B are said to be equal, if A is a subset of B and B is a subset of A. Thus the elements of set A are the same as the elements of set B, if the sets A and B are equal.

Definition 1.6. The Cartesian products of n sets $X_1 \times X_2 \times \cdots \times X_n$ is the set of ordered n-tuples, $X_1 \times X_2 \times \cdots \times X_n = \{\, (x_1, x_2, \ldots, x_n) : x_i \in X_i\,\}$, where $(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \iff x_i = y_i\ \forall\ i = 1, 2, \cdots n$

Definition 1.7. The power set $\mathbb{P}(X)$ of a set X is the set of all subsets of X.

For example, if $A = \{\, 1, 2\,\}$ then $\mathbb{P}(A) = \{\, \theta, \{\, 1\,\}, \{\, 2\,\}, \{\, 1, 2\,\}\,\}$. The power set of a finite set with x elements has 2^x elements. Also the power set of an infinite set, such as $\mathbb{N}$, consists of all finite and infinite subsets and it is infinite.

Definition 1.8. A set X is said to be *finite*, if X is empty (i.e $X = \emptyset$) or there is a bijection $f : X \mapsto \{\, 1, 2, \ldots n\,\}$; for some $n \in \mathbb{N}$. Otherwise it is called *infinite*.

Definition 1.9. A function $f : X \mapsto Y$ between sets X, Y assigns to each $x \in X$ a unique element $f(x) \in Y$.

A function can also be called maps, mapping or transformations. The set X on which f is defined is called the *domain* of f and the set Y in which it takes its values is called the *codomain*. Also the *range of* f *denoted as ranf* is the set of all possible values of $f(x)$ as x runs through the domain X of f; and it is generally a subset of the codomain Y.

We write $f : x \mapsto f(x)$ to indicate that f is the function that maps x to $f(x)$. For example, the identity function $id_x : X \mapsto X$ on a set X is the function $id_x : x \mapsto x$ that maps every element to itself.

Functions are classified in numerous ways, however we shall concentrate on some classifications which are important for the purpose of our work.

Definition 1.10. A function $f : X \mapsto Y$ is *injective* (or one to one) if $f(x_1) = f(x_2) \implies x_1 = x_2$

We call an injective function an *injection*. For example, the functions $f, g, h : \mathbb{R} \mapsto \mathbb{R}$ given by $f(x) = x$, $g(x) = x^3$ and $h(x) = e^x$ are all *injective*.

Proof:
We show that the functions defined above are all injectives.

First, we show that $f(x) = x$, $x \in \mathbb{R}$ is injective. Let $f(x_1) = f(x_2)\ \forall\ x \in \mathbb{R}$, since $f(x) = x$ we have that $x_1 = x_2$. Therefore $f(x_1) = f(x_2) \implies x_1 = x_2$. Hence f is injective. Second, we show that $g(x_1) = g(x_2)$ is injective. Let $g(x_1) = g(x_2)\ \forall\ x \in \mathbb{R}$, since $g(x) = x^3$ we have that $x_1^3 = x_2^3 \implies x_1 = x_2$. Therefore $g(x_1) = g(x_2) \implies x_1 = x_2$, hence g is injective.

Finally, let $h(x_1) = h(x_2)\ \forall\ x \in \mathbb{R}$, since $h(x) = e^x$ we have that $e^{x_1} = e^{x_2} \implies x_1 = x_2$. Therefore $h(x_1) = h(x_2) \implies x_1 = x_2$, hence h is injective.

While the functions $p, q, r : \mathbb{R} \mapsto \mathbb{R}$ given by $p(x) = 1$, $q(x) = x^2$, and $r(x) = \sin x$ are not *injective* since $p(0) = p(1)$ but $0 \neq 1$, $q(-1) = q(1)$ but $-1 \neq 1$, and $r(0) = r(\pi)$ but $0 \neq \pi$.

Definition 1.11. A function $f : X \mapsto Y$ is *surjective* (or onto) if for each $y \in Y$ we can find $x \in X : f(x) = y$.

A surjective function is called a *surjetion*. For example, the functions $f, g, h : \mathbb{R} \mapsto \mathbb{R}$ given by $f(x) = x$, $g(x) = x^3$, $h(x) = e^{x^2} \sin(x)$ are all *surjective*.
Proof:
We show that the functions defined above are all surjectives.
First, we show that $f(x) = x$, $x \in \mathbb{R}$ is surjective. It is clear that $\forall\ x \in \mathbb{R}\ \exists\ x \in \mathbb{R} :\ f(x) = x$.
Second, we show that $g(x) = x^3$, $x \in \mathbb{R}$ is surjective. It is also clear that $\forall\ x^3 \in \mathbb{R}\ \exists\ x \in \mathbb{R} : g(x) = x^3$.
Finally, we show that $h(x) = e^{x^2} \sin x$, $x \in \mathbb{R}$ is surjective. It is easy to see that $\forall\ e^{x^2} \sin x \in \mathbb{R}\ \exists\ x \in \mathbb{R} :\ h(x) = e^{x^2} \sin x$
While the functions $p, q, r : \mathbb{R} \mapsto \mathbb{R}$ given by $p(x) = 1$, $q(x) = e^x$ and $r(x) = \arctan(x)$ are not *surjective*.
Proof:
We prove that the above functions are not surjectives. Let $p(x) = 1$, $x \in \mathbb{R}$. Observe that $\nexists\ x \in \mathbb{R} :\ p(x) = 2$ but $2 \in \mathbb{R}$.
Second, let $q(x) = e^x$, $x \in \mathbb{R}$. Observe that $\nexists\ x \in \mathbb{R} :\ q(x) = 0$ but $0 \in \mathbb{R}$.
Finally, let $r(x) = \arctan(x)$, $x \in \mathbb{R}$. Observe that $\nexists\ x \in \mathbb{R} :\ r(x) = 90$ but $90 \in \mathbb{R}$.

Definition 1.12. A function $f : X \mapsto Y$ is *bijective* (or a one to one correspondence) if it is both *injective* and *surjective*.

A bijective function is called a *bijection*. For example, the identity function $id_x : X \mapsto X$ defined as $id_x(x) = x$, the function $f : \mathbb{R} \mapsto \mathbb{R}$ defined as $g(x) = x^3$ are all *bijective*.
Proof:
We prove that the above functions are bijectives. In the above examples we have shown that $id_x(x) = x$ and $g(x) = x^3$ are both injective and surjective, therefore the functions are bijectives.

Definition 1.13. let $f : X \mapsto Y$ be a *bijection*. we define $f^{-1} : Y \mapsto X$ by the rule $f^{-1}(y) = x \iff f(x) = y$; we call this the *inverse function* of f.

Definition 1.14. The composition of function $f : X \mapsto Y$ and $g : Y \mapsto Z$ is the function $g \circ f : X \mapsto Z$ define by $(g \circ f)(x) = g(f(x))$.

The order of application of the function in a composition is crucial and is read from right to left.

Remark 1.1. $f^{-1} \circ f = id_x$ and $f \circ f^{-1} = id_y$

Compostions: The composition of bijection is a bijection.

Injection: the restriction of an injection to a subset of its domain is still an injection

Inverse functions: The inverse function of a bijection is a bijection.

Definition 1.15. A set X is said to be *indexed by a set* I or equivalently, X is an indexed set if there is an onto function $f : I \mapsto X$. We then write $X = \{ x_i : i \in I \}$, where $x_i = f(i)$.

For example, $\{ 1, 4, 9, 16, \dots \} = \{ n^2 : n \in \mathbb{N} \}$. The set X itself is the range of the indexing function f.

Definition 1.16. let $C = \{ X_i : i \in I \}$ be an indexed collection of sets X_i; then we denote the *union* and *intersection* of sets in C by

$$\bigcup_{i \in I} X_i = \{ x : x \in X_i \text{ for some } i \in I \}, \qquad \bigcap_{i \in I} X_i = \{ x : x \in X_i \ \forall\ i \in I \}$$

For example, let $A_n = [\frac{1}{n}, 1 - \frac{1}{n}]$ for some $n \in \{ 3, 4, 5, \dots \}$ then $\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n=3}^{\infty} A_n = (0, 1)$. Also let $B_n = (-\frac{1}{n}, \frac{1}{n})$ for $n \in \{ 1, 2, 3, \dots \}$ then $\bigcap_{n \in \mathbb{N}} B_n = \bigcap_{n=1}^{\infty} B_n = \{ 0 \}$.

Definition 1.17. A set X is said to have a *cardinality* or *size* n, if there is a *bijection* $f : X \mapsto \{ 1, 2, 3, \dots, n \}$.

Two sets A and B are said to have *the same cardinality* (or *equivalent*) written as $|A| = |B|$ or $A \sim B$; if $\exists$ a bijection from A to B.

Proposition 1.1. *Two sets* having the same cardinality *defines an* equivalence relation *between sets.*

Proof:

$$|A| = |A| \qquad \text{(Reflexivity)}$$

The identity map $f(a) = a;\ \forall\ a \in A$ is a bijection from A to itself.

$$If \quad |A| = |B|, \text{ then } |B| = |A| \qquad \text{(Symmetry)}$$

Since $|A| = |B|$ then $\exists$ a bijection $f : A \mapsto B$, but the inverse function $f^{-1} : B \mapsto A$ is also a bijection (since the inverse function of a bijection is a bijection). Implying that $|B| = |A|$.
Finally,

$$If \quad |A| = |B| \text{ and } |B| = |C| \text{ then } |A| = |C| \qquad \text{(Transitivity)}$$

If $|A| = |B|$ then $\exists$ a bijection $f : A \mapsto B$, also if $|B| = |A|$ then $\exists$ a bijection $g : B \mapsto C$. But the composition of a bijection is again a bijection, it therefore follows that $g \circ f : A \mapsto C$ is again a bijection. Implying that $|A| = |C|$

Definition 1.18. A set X is said to be *countable* if :

- X is finite
  or
- there exists a *bijection* between X and the set of natural numbers $\mathbb{N}$.

2 Countability of Sets

In this section, we only focus on *the study of the major theorems in respect to countability of sets*. The following are some of the theorems:

2.1 Theorems

Theorem 2.1. *Any subset of* $\mathbb{N}$ *is countable*

Proof:

Without loss of generality, assume that A is an infinite subset of $\mathbb{N}$. Define a function $f : \mathbb{N} \mapsto A$ as follows let $f(1)$ be the smallest element of A (in the usual ordering of $\mathbb{N}$). This exists by well-ordering principle, since $A \neq \emptyset$. Then let $f(2)$ be the smallest element in $A\backslash\{ f(1) \}$. Note that this set is also non-empty (since A, being infinite, cannot be equal to $\{ f(1) \}$), so the well ordering principle applies again. In general, given $\{ f(1), f(2), \ldots, f(n) \}$, we let $f(n + 1)$ be the smallest element in $A\backslash\{ f(1), f(2), \ldots, f(n) \}$ (which is a non-empty subset of $\mathbb{N}$). This defines the function f inductively; f is *injective*, since from the construction we have : $f(1) < f(2) < f(3) < \cdots < f(n) < f(n + 1) < \cdots$
Next, we show that f is surjective, suppose for contradiction that f is not onto, assume that $A\backslash f(\mathbb{N}) \neq \emptyset$ and let a be the smallest element in this set. Thus $a - 1 = f(N)$ for some $N \in \mathbb{N}$. Then $f(N + 1)$ is the smallest element in $A\backslash\{ f(1), f(2), \ldots, f(n) \}$, so $f(N + 1) > a - 1$ (since $a - 1 = f(N)$ in this set). Thus $f(N + 1) > a$, but since $a \in A\backslash\{ f(1), f(2), \ldots, f(n) \}$ we can't have $f(N + 1) > a$ thus $f(N + 1) = a$, contradicting $a \notin f(\mathbb{N})$.

Corollary 2.1. *If* B *is countable and* $A \subset B$, $(A \neq \emptyset)$, *then* A *is countable*

Proof:

If B is finite, A is clearly finite. If B is countably infinite, there is a bijection $f : B \mapsto \mathbb{N}$. Then $f(A) \subset \mathbb{N}$, so by theorem 2.1; $f(A)$ is either finite or countably infinite. Since $A \sim f(A)$ (given that f is injective), it follows that A is countable.

Corollary 2.2. *If* A *is uncountable and* $A \subset B$, *then* B *is uncountable.*

Proof:

Suppose for contradiction that B is countable,
Case 1: If B is finite, then $A \subset B$ is a contradiction (since A is uncountable).
Case 2: If B is infinitely countable, then $\exists$ a bijection $f : B \mapsto \mathbb{N}$, it follows that $f : A \mapsto f(A)$ is also a bijection. But $f(A) \subset f(B) = \mathbb{N} \implies f(A) \subset \mathbb{N}$, therefore $f(A)$ is countable. Since there is a bijection from A to $f(A)$ it holds that $|A| = |f(A)|$, which is also a contradiction. (since an uncountable set can never be equivalent with countable set).

Corollary 2.3. *The intersection of finitely many countable sets is countable*

Proof:

Let $A_i, i = 1, 2, \ldots, n$ be countable sets for each i; then
$\bigcap_{i=1}^{n} A_i \subset A_i$ for each $i = 1, 2, \ldots, n.$ but, $A_i, i = 1, 2, \ldots, n.$ is countable for each i.
Hence, by theorem 2.1; $\bigcap_{i=1}^{n} A_i$, $i = 1, 2, \ldots, n.$ is countable.

Theorem 2.2. *If* $f : X \mapsto Y$ *is injective and* Y *is countable; then* X *is countable*

Proof:

If X is finite, then we have nothing to prove. So let X be infinite, now X is equivalent to $f(X)$ (since f is injective), where $f(X)$ is the range of f. So $f(X)$ is infinite. Also $f(X) \subseteq Y$, therefore Y is infinite. By hypothesis Y is countable so Y is countably infinite. By corollary 2.1 $f(X)$ is countable. Since $X \sim f(X)$. Hence X is countable.
See also [14]

Proposition 2.1. *Let X be a non-empty set. Then the following are equivalent*

1. *X is countable*
2. *There exists a surjective function $f : \mathbb{N} \mapsto X$*
3. *There exists an injective function $g : X \mapsto \mathbb{N}$*

Proof:

(1) $\implies$ (2). If X is countably infinite, then $\exists$ a bijection $f : \mathbb{N} \mapsto X$; then (2) follows. If X is finite; then there is a bijection $h : \{1, \ldots, n\} \mapsto X$ for some $n \in \mathbb{N}$. Then the function $f : \mathbb{N} \mapsto X$ defined by

$$f(i) = \begin{cases} h(i); & \text{if } 1 \leq i \leq n \\ h(n); & \text{if } i > n \end{cases}$$

is a surjection.
we show that the above function is surjective. Let $i \in \{1, 2, 3, \ldots, n\}$, then $f(i) = h(i)$, but by hypothesis $h : \{1, \ldots, n\} \mapsto X$ is a bijection. It therefore follows that $h(i)$ is a surjection and so is $f(i)$; since $f(i) = h(i)$.
Next, let $i \in \{n+1, n+2, n+3, \ldots, n+j, \ldots\}$, $j \in \mathbb{N}$ then $f(i) = h(n)$.
Without loss of generality, $h : \{1, \ldots, n\} \mapsto X$ is bijective $\implies$ $h(1) = k_1$, $h(2) = k_2$, $h(3) = k_3$, $\ldots$, $h(n-1) = k_{n-1}$, $h(n) = k_n$. Where $\{k_1, k_2, k_3, \ldots, k_n\} \in X$, $k_i \in \mathbb{R}$ for each $i \in \{1, 2, 3, \ldots, n\}$. So that $|X| = n$.
From definition of f, $f(i) = k_n$ for each $i > n \implies f(n+1) = k_n$, $f(n+2) = k_n$, $\ldots$, $f(n+j) = k_n$, $\ldots$
This implies that the function f has the same codomain as h , which is X. but $X = ranh = ranf$ $\implies$ the $ranf$ is the same as the coodomain. Hence f is a surjection.
(2) $\implies$ (3). let $f : \mathbb{N} \mapsto X$ be surjective. We claim that there is an injection $g : X \mapsto \mathbb{N}$. Given $x \in X$, the preimage $f^{-1}(\{x\}) \neq \emptyset$ (since f is surjective). By well-ordering principle, this set has a smallest element, we let $g(x)$ be this smallest element (i.e $g(x) = minf^{-1}(\{x\})$). g is injective since for two elements $x \neq x' \in X$ the preimages $f^{-1}(\{x\})$ and $f^{-1}(\{x'\})$ are disjoint (i.e $f^{-1}(\{x\}) \cap f^{-1}(\{x'\}) = \emptyset$) $\implies g(x) = minf^{-1}(\{x\}) \neq minf^{-1}(\{x'\}) = g(x')$ and hence their smallest elements are distinct.
(3) $\implies$ (1). Let $g : X \mapsto \mathbb{N}$ be an injective, we show that X is countable.
Since $g : X \mapsto g(X)$ is a bijection and $g(X) \subset \mathbb{N}$, hence X is countable.

Corollary 2.4. *If the function $f : X \mapsto Y$ is surjective and X is countable then Y is countable*

Proof:

By hypothesis, f is surjective. Therefore f has right-inverse $g : Y \mapsto X$, that is $f \circ g(y) = y$ $\forall$ $y \in Y$. The function g is injective since it has a left - inverse f, so by theorem 2.2 and from our hypothesis that X is countable we conclude that Y is countable.

Theorem 2.3. *A countable union of countable sets is countable*

Proof:

Consider sets $A_i = \{ a_{1i}, a_{2i}, a_{3i}, \ldots, \}, i = 1, 2, 3, \ldots$ where each A_i for $i = 1, 2, 3, \ldots$ is countable. The kth element of A_i is a_{ki}.
Now; it follows that

$$\bigcup_{i=1}^{\infty} A_i = \{ a_{11}, a_{12}, a_{21}, a_{13}, a_{22}, a_{31}, a_{14}, a_{23}, \ldots, a_{nm}, \ldots \}$$

Note that the order has been taken according to the sum $m + n = l$ where $l = 2, 3, \ldots, n, m$ being the suffices of the element $a_{nm} \in A_i$
See also [14]

let

$$\bigcup_{i=1}^{\infty} A_i = \{ x_1, x_2, x_3, x_4, x_5, \ldots, x_n, \ldots \}$$

where

$$\bigcup_{i=1}^{\infty} A_i = \{ \underbrace{a_{11}}_{x_1}, \underbrace{a_{12}}_{x_2}, \underbrace{a_{21}}_{x_3}, \underbrace{a_{13}}_{x_4}, \underbrace{a_{22}}_{x_5}, \underbrace{a_{31}}_{x_6}, \underbrace{a_{14}}_{x_7}, \underbrace{a_{23}}_{x_8}, \ldots, \underbrace{a_{nm}}_{x_n}, \ldots \}$$

then, the function $f : \bigcup_{i=1}^{\infty} A_i \mapsto \mathbb{N}$ defined as $f(x_i) = i$; $i \in \{ 1, 2, 3, \ldots \}$ is a bijection between the elements of $\bigcup_{i=1}^{\infty} A_i$ and $\mathbb{N}$, the set of natural numbers. Now, we show that the function defined above is injective. Suppose for contradiction that $f(x_i) = i$; $i = 1, 2, 3, \ldots$ is not injective; then $f(x_j) = f(x_i) \implies x_i \neq x_j$ for some $i, j \in \mathbb{N}$. But $f(x_i) = i$ and $f(x_j) = j$. Also, $f(x_j) = f(x_i) \implies i = j$. Let $i = j := j^*$, then $x_j = x_{j^*}$; $x_i = x_{j^*}$. From our assumption we have that $x_j \neq x_i = x_{j^*} = x_j \implies x_j \neq x_j$ which is a contradiction, hence f is injective.
Next, the function f is surjective since the codomain of f is equal to its range. Hence, the function $f : \bigcup_{i=1}^{\infty} A_i \mapsto \mathbb{N}$ define as $f(x_i) = i$; $i \in \{ 1, 2, 3, \ldots \}$ is a bijection. Therefore, the set $\bigcup_{i=1}^{\infty} A_i$ is countable.

Theorem 2.4. *The Cartesian product of finitely many countable sets is countable.*

Proof:

We prove this theorem by induction. Let $p(n)$ be a statement that depends on our theorem (i.e if A_i is a countable set for each $i \in \{ 1, 2, \ldots n \}$ then, $A_1 \times A_2 \times \cdots \times A_n$ is countable), let $n = 2$, then we show that $A_1 \times A_2$ is countable: if any of the two sets is empty then $A_1 \times A_2 = \emptyset$ and we have nothing to prove. If one of the sets is finite, say A is finite with k elements, then the product of $A_1 = \{ a_1, a_2, \ldots, a_k \}$ and $A_2 = \{ b_1, b_2, \ldots, b_n, \ldots \}$ is

$$\begin{array}{rllll} A_1 \times A_2 = \{ & (a_1, b_1), & (a_1, b_2), & \ldots, & (a_1, b_n), \ldots \\ & (a_2, b_1), & (a_2, b_2), & \ldots, & (a_2, b_n), \ldots \\ & \vdots & \vdots & \vdots & \vdots \\ & (a_k, b_1), & (a_k, b_2), & \ldots, & (a_k, b_n), \ldots \} \end{array}$$

can be seen to be equivalent to $\mathbb{N}$ by listing the elements as
$\{ (a_1, b_1), (a_2, b_1), \ldots, (a_k, b_1); (a_1, b_2), (a_2, b_2), \ldots, (a_k, b_2); \ldots; (a_1, b_n), (a_2, b_n),$
$\ldots, (a_k, b_n); \ldots \}$
Next, let A and B be both countably infinite: $A = \{ a_1, a, \ldots \}$, $B = \{ b_1, b_2, \ldots \}$. Then $A \times B$ is equivalent to $\mathbb{N}$ can be exhibited as

$$\begin{array}{rllll} A_1 \times A_2 = \{\,(a_1,b_1), & (a_1,b_2), & \ldots, & (a_1,b_n),\ldots & \\ (a_2,b_1), & (a_2,b_2), & \ldots, & (a_2,b_n),\ldots & \\ \vdots & \vdots & \vdots & \vdots & \\ (a_k,b_1), & (a_k,b_2), & \ldots, & (a_k,b_n),\ldots & \\ \vdots & \vdots & \vdots & \vdots & \ldots\} \end{array}$$

putting or arranging in order, we have:
$A_1 \times A_2 = \{\,(a_1,b_1);(a_2,b_1),(a_1,b_2);(a_3,b_1),(a_2,b_2),(a_1,b_3);(a_4,b_1),(a_3,b_2),$
$(a_2,b_3),(a_1,b_4);\ldots\}$
The function $f : A_1 \times A_2 \mapsto \mathbb{N}$ define as $f(a_1,b_1) = 1$, $f(a_2,b_1) = 2$, $f(a_1,b_2) = 3$, $f(a_3,b_1) = 4$,... is a bijection; hence $A_1 \times A_2$ is countably infinite. Therefore $p(2)$ is true.
See also [14]
Assume that the statement is true for $p(n-1)$, that is $A_1 \times A_2 \times \cdots \times A_{n-1}$ is countable.
We now move further to prove that $p(n)$ is true $\forall\ n \in \mathbb{N}$. That is we show that $A_1 \times A_2 \times \cdots \times A_n$ is countable.
let $A_1 \times A_2 \times \cdots \times A_{n-1} = K$, we now show that $\underbrace{A_1 \times A_2 \times \cdots \times A_{n-1}}_{K} \times A_n$ is countable. But this reduces to only showing that $K \times A_n$ is countable. K is countable by induction assumption and A_n is also countable by hypothesis. Hence $K \times A_n$ is countable, which has already been established in the first step of our proof (that is $p(2)$ is true). Implying that $p(n)$ is true if $p(n-1)$ is true $\forall$ $i \in \{\,1,2,\ldots n\,\}$. Therefore the above theorem is true.

Theorem 2.5. *There is no surjection from a set A to $\mathbb{P}(A)$.*

Proof:

Consider any function $f : A \mapsto \mathbb{P}(A)$ and let $B = \{\,a \in A \mid a \notin f(a)\,\}$. We claim that there is no $b \in A : f(b) = B$. Indeed, assume $f(b) = B$ for some $b \in A$, then either $b \in B$ hence $b \notin f(b)$ which is a contradiction or $b \notin B = f(b)$ implying that $b \in B$ which is again a contradiction. Hence the map f is not surjective as claimed.

3 Applications of Theorems on Sets

In these section, we will show the applications of the theorems studied in the previous section.

3.1 Examples

Example 3.1. *Every finite set is countable*

Proof:

This follows from the definition of *countable sets*

Example 3.2. *The set of all integers $\mathbb{Z}$ is countable*

Proof:

Let $f : \mathbb{N} \mapsto \mathbb{Z}$ be define as:

$$f(n) = \begin{cases} \frac{n}{2}; & \text{if } n \text{ is even} \\ \frac{1-n}{2}; & \text{if } n \text{ is odd} \end{cases}$$

See also [13]
It suffice to show that $f(n)$ define above is a bijection. We progress as follows: Observe that

$f(n_1) = f(n_2) \implies \frac{n_1}{2} = \frac{n_2}{2}\ \forall\ n_1, n_2$ even. So $n_1 = n_2$. Hence $f(n_1) = f(n_2) \implies n_1 = n_2\ \forall\ n$ even. Also let $f(n_1) = f(n_2) \implies \frac{1-n_1}{2} = \frac{1-n_2}{2};\ \forall\ n_1, n_2$ odd $\implies 1 - n_1 = 1 - n_2 \implies n_1 = n_2$. Hence $f(n_1) = f(n_2) \implies n_1 = n_2\ \forall\ n$ odd. Therefore, f is injective. Next, $\forall\ \frac{n}{2}, \frac{1-n}{2} \in \mathbb{Z}, \exists\ n \in \mathbb{N} : f(n) = \frac{n}{2}$ and $f(n) = \frac{1-n}{2}$. Hence f is surjective. In conclusion, f is a bijection, Implying that $\mathbb{Z}$ is countable.

Example 3.3. *The set of all rational numbers is countable*

Proof:

Let the set of all rational numbers be denoted as $\bigcup_{i=1}^{\infty} A_i$, where A_i is the set of rational numbers which can be written with denominator i. Let such sets be $A_i = \left\{ \frac{0}{i}, \frac{-1}{i}, \frac{1}{i}, \frac{-2}{i}, \frac{2}{i}, \dots \right\}$, $i \in \{1, 2, \dots\}$. But each A_i is equivalent to the set of all positive integers and by theorem 2.3, countable. See also [14]

Example 3.4. *The set* $\mathbb{R}$ *of real numbers is uncountable*

Proof:

Suppose for contradiction that the set $\mathbb{R}$ is countable. Then $\mathbb{R} = \{x_1, x_2, x_3, x_4, \dots\}$. Enclose each member x_n of $\mathbb{R}$ in an open interval $I_n = \left(x_n - \frac{1}{2^{n+1}}, x_n + \frac{1}{2^{n+1}}\right)$ of length $\frac{1}{2^n}$ (i.e $L(I_n) = x_n + \frac{1}{2^{n+1}} - x_n - \frac{1}{2^{n+1}} = \frac{1}{2^n}$), $n = 1, 2, 3, \dots$ The sum of the lengths of I_n's is $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$ (that is sum to infinity of a geometric progression). But $x_n \in \mathbb{R}$ and $\mathbb{R} = \bigcup_n^{\infty} \{x_n\} \subseteq \bigcup_n^{\infty} I_n$ implies that the whole real line (whose length is infinite) is contained in the union of intervals whose lengths add up to 1. Which is a contradiction, hence $\mathbb{R}$ is uncountable.
See also [14]

Example 3.5. *The set* $\mathbb{P}(\mathbb{N})$ *is uncountable*

Proof:

By theorem 2.5 and corollary 2.4 we get that $\mathbb{P}(\mathbb{N})$ is uncountable.

Example 3.6. *The set* $\mathbb{N} \times \mathbb{N}$ *is countable*

Proof:

By proposition 2.1; it suffices to construct an injective function $f : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{N}$. Let $f : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{N}$ be define as $f(a, b) = 2^a 3^b$. Assume that $2^a 3^b = 2^x 3^y$. If $a < x$ then $3^b = 2^{x-a} 3^y$.The left side of this equality is an odd number whereas the right side of the equation is an even number, which implies that $x = a$ and $3^b = 3^y$. Hence $b = y$, therefore f is injective. Therfore by theorem 2.2, the set $\mathbb{N} \times \mathbb{N}$ is countable.

Example 3.7. *The set of real numbers in* $[0, 1]$ *is uncountable.*

Proof:

Let the set of all real numbers in $[0, 1]$ be countable, that is $\{x : 0 \leq x \leq 1\} = \{x_1, x_2, \dots, x_n, \dots\}$. Each real numbers in $[0, 1]$ has a decimal expansion 0, a_1, a_2,...,a_n,... where a_i, $i \in \mathbb{N}$, are any of the digits 0, 1, 2, ...,9. We assume that the numbers whose decimal expansion terminate such as 0.0573 are written as 0.0573000... which is the same as 0.0572999..., since all real numbers in $[0, 1]$ are countable, therefore, we can establish a one to one correspondence of the members of $[0, 1]$ with the set of positive integers in the following manner:
$1 \leftrightarrow 0.a_{11}a_{12}a_{13}\dots$
$2 \leftrightarrow 0.a_{21}a_{22}a_{23}\dots$
$3 \leftrightarrow 0.a_{31}a_{32}a_{33}\dots$

$4 \leftrightarrow 0.a_{41}a_{42}a_{43}\ldots$
....
We now construct a number $0.b_1b_2b_3\ldots$, where

$$b_i = \begin{cases} 4; & \text{if } a_{ii} = 5; \\ 5; & \text{if } a_{ii} \neq 5; \qquad i = 1, 2, 3, \ldots \end{cases}$$

(any two digits can be used instead of 4 and 5). Then the number $0.b_1b_2b_3\ldots$, lies between 0 and 1 and is different from the numbers in the above list and therefore cannot be in the list, contradicting the assumption that the set of all real numbers in $[0, 1]$ is countable.
See also [14]

Example 3.8. *The set of rational numbers in* $[0, 1]$ *is countable.*

Proof:

In order to show that the set of rational numbers in $[0, 1]$ is countable, we must show that there exists a one to one correspondence between the set of rationals of $[0, 1]$ and the set of natural numbers $\mathbb{N}$.
Arrange the set of rationals according to increasing denominators as : 0, 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, $\frac{1}{6}$, $\frac{5}{6}$, $\frac{1}{7}$, $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$, $\frac{6}{7}$, ... e.t.c. Then the one to one correspondence can be indicated as:

$1 \leftrightarrow 0$	$5 \leftrightarrow \frac{2}{3}$	$9 \leftrightarrow \frac{2}{5}$
$2 \leftrightarrow 1$	$6 \leftrightarrow \frac{1}{4}$	$10 \leftrightarrow \frac{3}{5}$
$3 \leftrightarrow \frac{1}{2}$	$7 \leftrightarrow \frac{3}{4}$	$11 \leftrightarrow \frac{4}{5}$
$4 \leftrightarrow \frac{1}{3}$	$8 \leftrightarrow \frac{1}{5}$	....

See also [14]

4 Conclusion

The authors studied the major theorems concerning countable sets and showed their applications on sets.

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Authors have declared that no competing interests exist.

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