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# An Implicit Method for Numerical Solution of System of First-Order Singular Initial Value Problems

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***Authors' contributions***

*This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.*

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## Abstract

Recently an implicit method has been developed for solving singular initial value problems numerically which have an initial singular point. The method is simple and gives significantly better results than the implicit Euler method as well as second order implicit Runge-Kutta (RK2) method. In this article, the system of first order singular initial value problems having an initial singular point has been solved by this method.

*Keywords: Systems of singular initial value problems; implicit Euler method; implicit Runge-Kutta method.*

**Mathematical Subject Classification (2010):** 35F40, 35F55.

## 1 Introduction

There are many mathematical models in physics, chemistry, and mechanics who take the form of systems of time-dependent partial differential equations subject to initial or boundary condition. For the investigation of stationary solutions, many of these models can be reduced to singular systems of ordinary differential equations when symmetries problems in the geometry have been used.

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Consider a system of first-order singular initial value problem in the form as

$$\begin{aligned} & z'(x) = \frac{M(x)}{x} z(x) + f(x, z(x)), \quad x \in (0, 1], \\ & B_0 z(0) = \beta, \\ & z \in C[0, 1], \end{aligned} \tag{1}$$

Where $z$ and $f$ are vector-valued functions of dimension $n$, $M$ is a $n \times n$ matrix, $B_0$ is a $m \times n$ matrix and $\beta$ is a vector of dimension $m \leq n$. Singular initial value problems are encountered in ecology in the computation of avalanche run-up [1]. Singular systems also arise in many areas of science and engineering problems such as constrained mechanical systems, fluid dynamics, chemical reaction kinetics, simulation of electrical networks, electrical circuit theory [2], etc. Several authors evaluated this system analytically as well as numerically. Koch et al. [3-4] discussed the existence of an analytic solution of this system. Sekar and Vijayarakavan [5] investigated the numerical solution of first order linear singular systems using Leapfrog method. Recently, Komashynska et al. [6] applied the residual-power series method (RPSM) to obtain efficient analytical solutions of this system.

For the numerical solution of the Eq. (1), various schemes such as explicit Runge-Kutta methods, multi-step methods have been applied. However, many high-order methods show order reductions when applied to singular problems. Explicit Runge-Kutta methods show a reduction down to order 2 in general [7], and multi-step methods deviate from their classical convergence order by a logarithmic term [8]. A basic low order method and then an acceleration technique also applied to obtain high accuracy numerical solution. Auzinger et al. [9] and Koch et al. [10,11] applied well-known acceleration technique Iterated Defect Correction (IDeC) based on implicit Euler method to obtain high accuracy. Low order Implicit Runge-Kutta method, e.g., second order implicit Runge-Kutta (RK2) method shows better approximation than implicit Euler method, but the results near the singular point are not significantly improved.

In this article, the present method has been utilized to solve the Eq. (1) having an initial singular point.

# 2 Methodology

Earlier, Huq et al. [12] derived a numerical integration formula for evaluating definite integral having an initial singular point, i.e., at $x = x_0$ as is given in Eq. (2)

$$\int_{x_0}^{x_0+3h} f(x)\, dx = \frac{3h}{4}\left[3 f(x_0 + h) + f(x_0 + 3h)\right] \tag{2}$$

Based on formula (2), an implicit method was derived by Hasan et al. [13] for solving first order initial value problem having an initial singular point.

Consider a first order initial value problem having an initial singular point, i.e., at $x = x_0$ is

$$y'(x) = f(x, y), \quad y(x_0) = y_0 \tag{3}$$

According to the formula (2), the first and others steps solutions of Eq. (3) are given in Eqs. (4a) and (4b)

$$y_1 = y_0 + \frac{3h}{4}\left[3f(x_0 + h,\ y_0) + f(x_0 + 3h, y_1)\right] \tag{4a}$$

$$y_{i+1} = y_{i-1} + \frac{3h}{4}\left[3f(x_i,\ y_i) + f(x_{i+1}, y_{i+1})\right],\ for\ \ i \geq 1 \tag{4b}$$

where, $x_{i+1} = x_i + 2^i.3h$ .

Since the formula (i.e., equations (4a) and 4(b)) was derived for the unequal interval as $h, 3h, 2.3h, 2^2.3h, 2^3.3h, \cdots$. In this regard step size as well as error gradually increased. To avoid this difficulty, the formulas (4a) and (4b) have been modified by Hasan et al. [14] as

$$y_{i+1} = y_i + \frac{h}{4}\left[3f(x_i + h/3, (y_i + (y_{i+1} - y_i)/3)) + f(x_{i+1}, y_{i+1})\right];\ i = 0, 1, 2, .... \tag{5}$$

where, $x_{i+1} = x_i + h$ . The Eq. (5) is the present formula for solving singular initial value problems.

Recently, Hasan et al. [15] extended this formula (i.e., using Eq. (5)) for solving second orders singular initial value problems having an initial singular point. In this article, some system of first-order peculiar initial value problems having an initial singular point has been solved by applying this method.

By particular choosing $M$ and $f$ , The Eq. (1) can be written as

$$\begin{aligned} z'(x) &= \begin{pmatrix} z_1'(x) \\ z_2'(x) \end{pmatrix} = \frac{1}{x}\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix} + x\begin{pmatrix} 0 \\ 4z_1(x) - 2 \end{pmatrix},\ \ x(0, 1] \\ z(0) &= \begin{pmatrix} 1/2 + 10/\sinh(2) \\ 0 \end{pmatrix} \end{aligned} \tag{6}$$

The equation (6) can be transformed into two first-order singular initial value problems as

$$z_1'(x) = y = \frac{z_2}{x} \tag{7a}$$

$$z_2'(x) = z = -\frac{z_2}{x} + 4x\,z_1 - 2x \tag{7b}$$

where $z_1(0) = 1/2 + 10/\sinh(2)$ and $z_2(0) = 0$ .

The approximate solutions of the equations (7a) and (7b) can be obtained by applying the present formula is given in Eq. (5) as

$$y_{i+1} = y_i + \frac{h}{4}\left[\frac{3(z_i + (z_{i+1} - z_i)/3)}{(x_i + h/3)} + \frac{z_{i+1}}{(x_i + h)}\right];\ \ i = 0, 1, 2, ... \tag{8a}$$

$$z_{i+1} = z_i + \frac{h}{4}\left[3\left\{-\frac{(z_i + (z_{i+1} - z_i)/3)}{(x_i + h/3)} + 4\,(x_i + h/3)\,(y_i + (y_{i+1} - y_i)/3) - 2\,(x_i + h/3)\right\} + \left\{-\frac{(z_{i+1})}{(x_i + h)} + 4(x_i + h)\,(y_{i+1}) - 2(x_i + h)\right\}\right] \tag{8b}$$

It is obvious that equations (8a) and (8b) are systems of equations for two unknown $y_{i+1}$ and $z_{i+1}$. These values calculated by Newton-Raphson method. To compare the present method to other classical methods such as the second order implicit Runge-Kutta (RK2) method and the implicit Euler method [16] are given in equations (9) and (10) respectively.

$$y_{i+1} = y_i + k\,; \qquad i = 0, 1, 2, ... \tag{9}$$

where $k = h\,f(x_i + h/2,\ y_i + k/2)$

and

$$y_{i+1} = y_i + h\,f(x_i + h,\ y_{i+1})\,; \qquad i = 0, 1, 2, ... \tag{10}$$

# 3 Convergence and Stability of the Present Method

The order of convergence of the present method (i.e., Eq. (5)) is $O(h^3)$, *i.e.*, the truncation error is $O(h^4)$. The truncation error of the second order implicit Runge-Kutta (RK2) method (i.e., Eq. (9)) and implicit Euler method (i.e., Eq. (10)) are $O(h^3)$ and $O(h^2)$ respectively.

To test the stability, consider a scalar test equation.

$$y' = \lambda\, y, \quad \lambda \in C, \quad \mathrm{Re}(\lambda) < 0 \tag{11}$$

Appling (5) to the test equation with $y' = f(x, y) = \lambda\, y$ gives

$$y_{i+1} = y_i + \frac{h}{4}[3\lambda\,(y_i + (y_{i+1} - y_i)/3) + \lambda\, y_{i+1}] \tag{12}$$

Solving Eq. (12) for $y_{i+1}$ and then substituting $z = \lambda\, h$, gives

$$y_{i+1} = \frac{(1 + z/2)}{(1 - z/2)}\, y_i = R(z) y_i \tag{13}$$

where, $R(z) = (1 + z/2)/(1 - z/2)$ is the stability function of the present method.

For $\lambda < 0$, then $|R(z)| < 1$ for any $h > 0$. Since $z$ is imaginary, the present method is absolutely stable in the entire negative half of the complex $z$ plane. The region of absolute stability is the set of all complex $z$ where $|R(z)| \le 1$, While $R(z)$ is a polynomial for an explicit method and it is a rational function for an implicit method [17]. A Runge-Kutta method is said to be A-stable [18] if its stability region contains $C^{-}$, the non-positive half-plane $\{z = \lambda\, h \in C : \mathrm{Re}(z) < 0\}$. So the present method is A-stable. The stability region of the present method is given in Fig. 1.

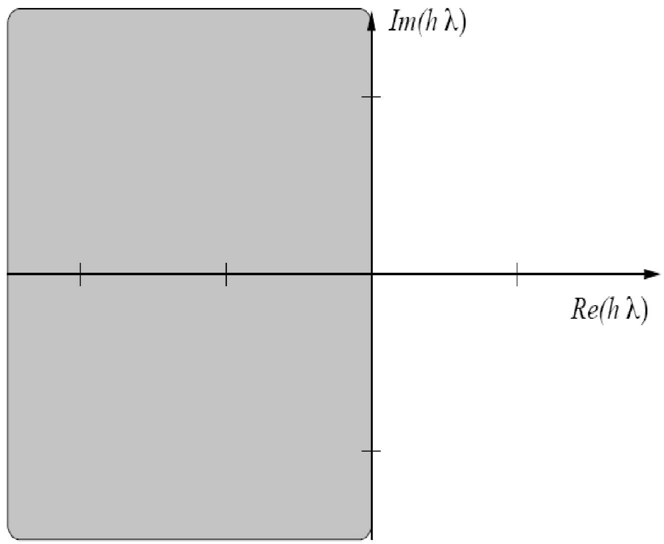


**Fig. 1. Stability region of the present method**

# 4 Application Examples

In this section some system of first-order singular initial value problems have been solved by the present method (*i.e.,* using Eq. (5)) and compare absolute errors among second order implicit Runge-Kutta (RK2) method (*i.e.,* using Eq. (9)) and the implicit Euler (*i.e.,* using Eq. (10)) method.

**Example 4.1**

Consider a system of first-order linear singular initial value problems (Auzinger et al. [9])

$$\begin{aligned} z'(x) &= \frac{1}{x}\begin{pmatrix} 0 & 1 \\ -15 & -8 \end{pmatrix} z(x) + e^{2x} x \begin{pmatrix} 0 \\ 4x^2 + 26x + 35 \end{pmatrix}, \; x(0,1] \\ z(0) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned} \tag{14}$$

With the exact result $(z_1(x), z_2(x)) = (x^2 e^{2x}, 2(x+1)x^2 e^{2x})$. The absolute errors of the first equation of the systems given in Eq. (14) obtained by the implicit Euler method and RK2 method are plotted in Fig. 2(a), the RK2 method and the present method are plotted in Fig. 2(b) for $h = 0.01$. Also the second equation of the systems given in Eq. (14) obtained by the implicit Euler method and RK2 method are plotted in Fig. 2(c), the RK2 method and the present method are plotted in Fig. 2(d) for $h = 0.01$. Figs. 2(a) and 2(c) shows that the error of the systems of the equation given in Eq. (14) obtained by Euler method much higher than RK2 method and also show in Figs. 2(b) and 2(d) that the error of the present method smaller than RK2 method.

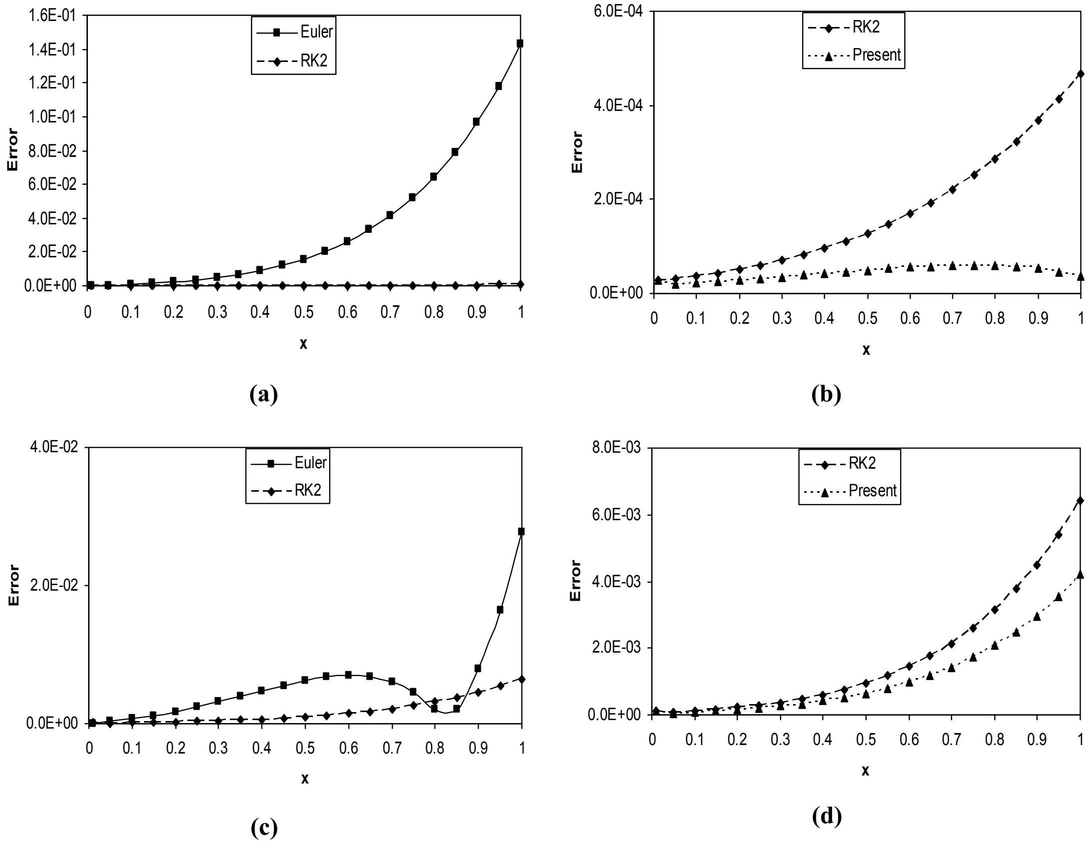


**Fig. 2. The absolute error of Eq. (14) for various methods having step size** $h = 0.01$

**Example 4.2**

Consider a system of first-order linear singular initial value problems (Auzinger et al. [9])

$$\begin{aligned} z'(x) &= \frac{1}{x}\begin{pmatrix} 0 & 1 \\ -2-8x+x^2 & -3 \end{pmatrix} z(x) + x\begin{pmatrix} 0 \\ 12e^{1-x} \end{pmatrix}, \quad x(0,1] \\ z(0) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned} \tag{15}$$

With the exact result $(z_1(x), z_2(x)) = (x^2 e^{1-x}, (2-x)x^2 e^{1-x})$. The absolute errors of the first equation of the systems given in Eq. (15) obtained by the implicit Euler method and RK2 method are plotted in Fig. 3(a), the RK2 method and the present method are plotted in Fig. 3(b) for $h = 0.01$. Also the second equation of the systems given in Eq. (15) Obtained by the implicit Euler method and RK2 method are plotted in Fig. 3(c), the RK2 method and the present method are plotted in Fig. 3(d) for $h = 0.01$. Figs. 3(a) and 3(c) shows that the error of the systems given in Eq. (15) Obtains by Euler method much higher than RK2 method. Fig. 3(b) shows that the error of RK2 method is higher than the present method for $0.1 < x \le 0.6$ and closer to other value $x$. However, Fig. 3(d) indicates that error of the present method lower than RK2.

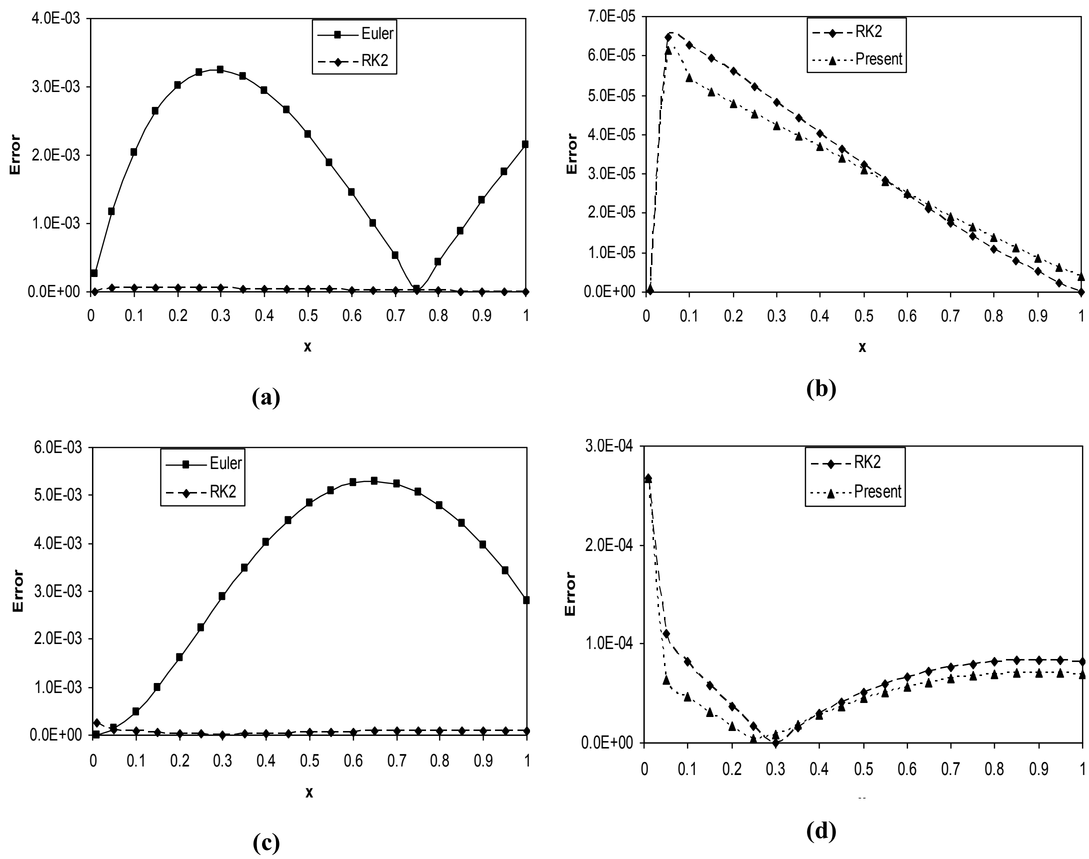


**Fig. 3. The absolute error of Eq. (15) for various methods having step size $h = 0.01$**

**Example 4.3**

Consider a system of first-order linear singular initial value problems (Auzinger et al. [9])

$$z'(x) = \frac{1}{x}\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} z(x) + x\begin{pmatrix} 0 \\ -9\cos(3x) - \frac{6}{x}\sin(3x) \end{pmatrix}, \; x(0,1] \qquad (16)$$

$$z(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

With the exact result $(z_1(x), z_2(x)) = (\cos(3x), -3x\sin(3x))$ .The absolute errors of the first equation of the systems given in Eq. (16) obtained by the implicit Euler method and RK2 method are plotted in Fig. 4(a), the RK2 method and the present method are plotted in Fig. 4(b) for $h = 0.01$ . Also the second equation of the systems given in Eq. (16) obtained by the implicit Euler method and RK2 method are plotted in Fig. 4(c), the RK2 method and the present method are plotted in Fig. 4(d) for $h = 0.01$ . Figs. 4(a) and 4(c) shows that the error of the systems given in Eq. (16) obtained by Euler method much higher than RK2 method. Fig. 4(b) shows that error of RK2 is higher than the present method. However Fig. 4(d) indicates that error of RK2 method is higher than for $x \le 0.35$ and $x > 0.55$ smaller than to the present method for $0.35 < x \le 0.55$ .

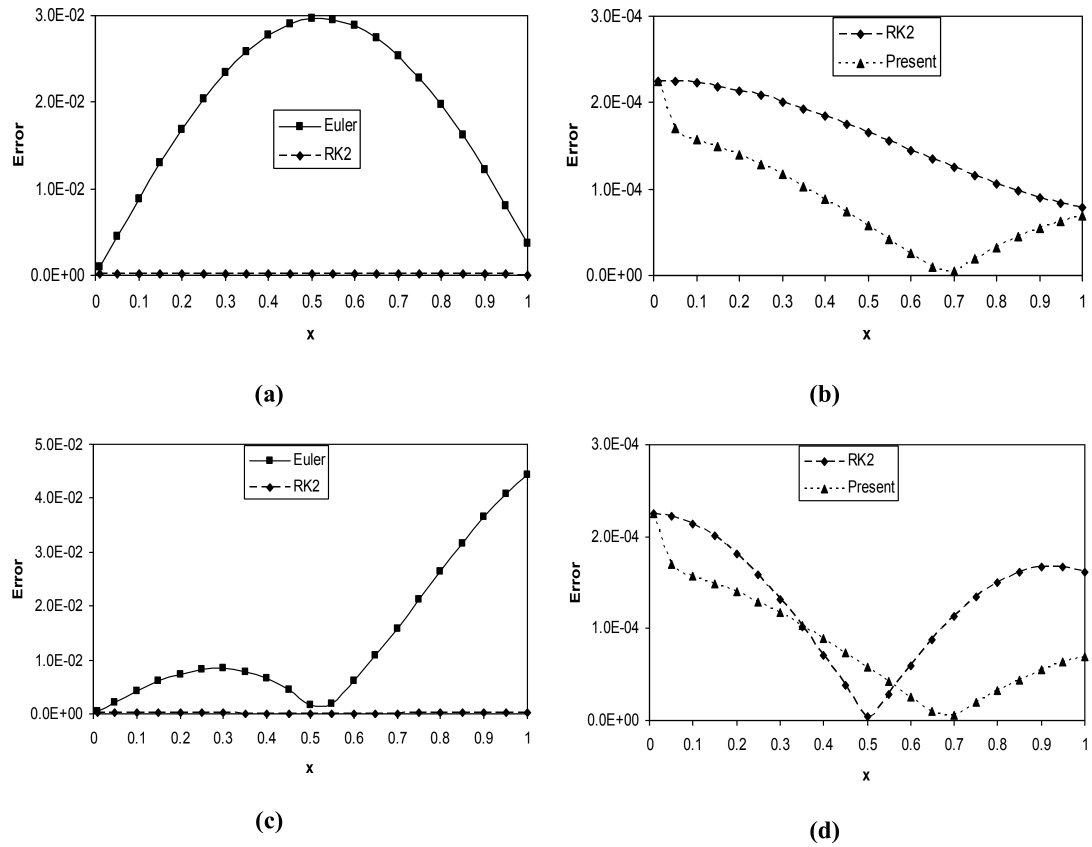


**Fig. 4. The absolute error of Eq. (16) for various methods having step size**
$h = 0.01$

**Example 4.4**

Consider a system of first-order nonlinear singular initial value problems (Auzinger et al. [9])

$$z'(x) = \frac{1}{x}\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} z(x) - x\begin{pmatrix} 0 \\ {z_1}^5(x) \end{pmatrix}, \quad x(0, 1]$$
$$z(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{17}$$

With the exact result $(z_1(x), z_2(x)) = (1/\sqrt{(1+x^2/3)}, -x^2/3\sqrt{(1+x^2/3)^3})$ . The absolute errors of the first equation of the systems given in Eq. (17) obtained by the implicit Euler method and RK2 method; The RK2 and the present method are plotted in Fig. 5(a) and Fig. 5(b) respectively for $h = 0.01$ . Also the second equation of the systems given in Eq. (17) obtained by the implicit Euler method and RK2 method; The RK2 and the present method are plotted in Fig. 5(c) and Fig. 5(d) respectively for $h = 0.01$ . Figs. 5(a) and 5(c) shows that the error of the systems given in Eq. (17) obtained by Euler method much higher than RK2 method. However Fig. 5(b) and 5(d) show that the error of RK2 method is much higher than the present method.

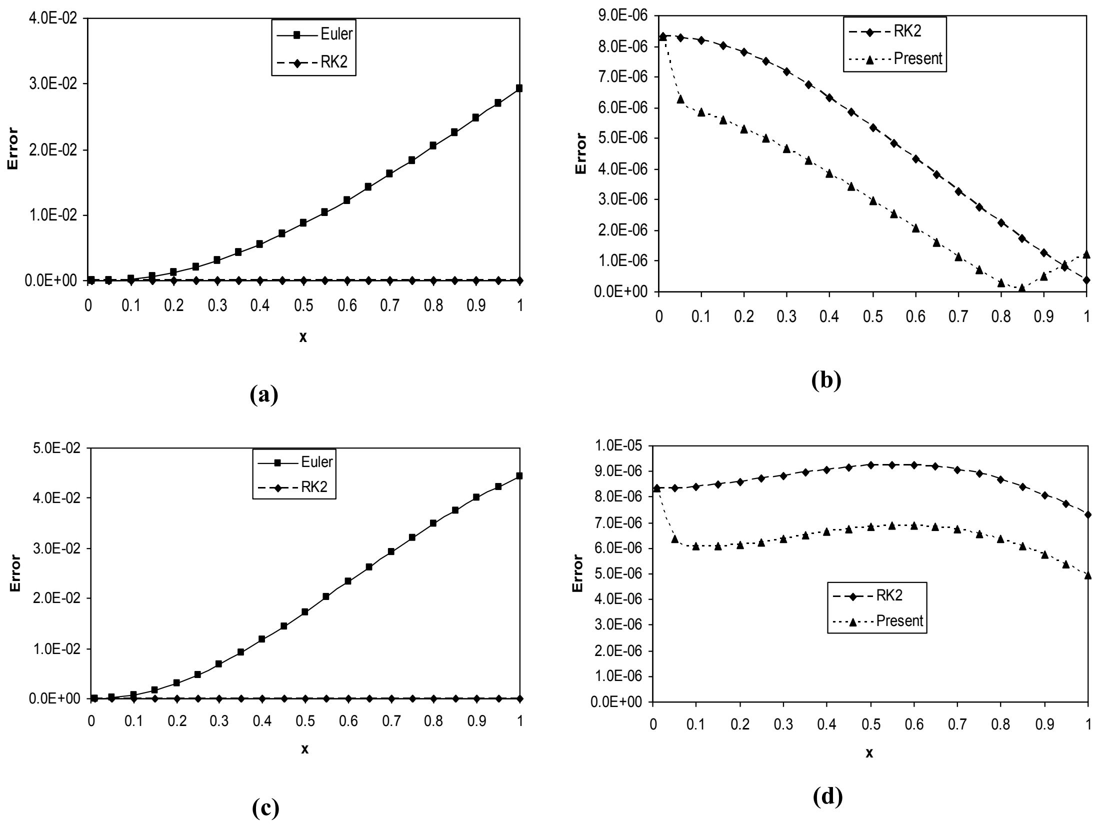


**Fig. 5. The absolute error of Eq. (17) for various methods having step size** $h = 0.01$

# 5 Results and Discussion

The various types of systems of first-order singular initial value problems have been solved by the present method. The variations of absolute error concerning for Euler, RK2 and the present method have been presented in Figs. 2- 5.

From the above Figs. 2- 5, it is observed that the error of the RK method is smaller than the Euler method. It is also observed that the error the present method is less than RK method. However, the RK method can produce less error than the present method in some region of the given interval (see Fig. 3(b) and Fig. 4(d)).

# 6 Conclusion

Overall the present method can show less than the RK method because the present method has a consistently lower error. Therefore, it is evident that the present method is more suitable than RK2 method as well as implicit Euler method for solving the system of first-order singular initial value problems.

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## Competing Interests

Authors have declared that no competing interests exist.

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