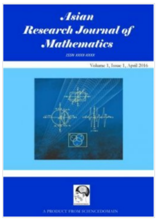

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Decay for Solutions to Semilinear Regularity-Loss Type Equations with Memory

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Authors' contributions

This work was carried out in collaboration between both authors. Author SM designed the study and performed the analysis. Author LW performed the computation and proof of the study. Both authors read and approved the final manuscript.

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Abstract

In this paper we consider the initial value problem of an inertial model for a generalized semilinear plate equation with memory in $\mathbb{R}^n$ $(n \geq 1)$. We study the decay and the regularity-loss property for this type of equations in the spirit of [1, 2]. The novelty of this paper is that we extend the order of derivatives from integer to fraction and refine the results in the related literature [1, 3].

Keywords: Plate equation; decay; regularity-loss; memory.

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1 Introduction

In this paper we consider the initial value problem of an inertial model for the following generalized semilinear plate type equation with memory in $\mathbb{R}^n$ ($n \geq 1$):

$$\begin{cases} u_{tt} - \Delta u_{tt} + (-\Delta)^p u + u + g * \Delta u = f(u, u_t, \nabla u), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x). \end{cases} \tag{1.1}$$

Here $p \geq 1$ is a real number, the subscript t in u_t and u_{tt} denotes the time derivative (i.e., $u_t = \partial_t u$ and $u_{tt} = \partial_t^2 u$), $u = u(x,t)$ is the unknown function of $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$, and Δu_{tt} corresponds to the rotational inertia. The memory term $g * \Delta u := \int_0^t g(t-\tau)\Delta u(\tau) d\tau$ means that the stress at an instant depends on the whole history of the strains the material has suffered. We assume the following assumptions:

Assumption [A]: $g \in C^2(\mathbb{R}^+)$, $g(s) > 0$, and there exist $C_i > 0$ $(i = 0, 1, 2)$ such that

i) $-C_0 g(s) \leq g'(s) \leq -C_1 g(s)$, $|g''(s)| \leq C_2 g(s)$, $\forall s \in \mathbb{R}^+$,

ii) $\int_0^\infty g(s) ds < 1$.

Assumption [B]: $f \in C^\infty(\mathbb{R}^{n+2})$ and there exists $\alpha \in \mathbb{Z}^+$ satisfying $\alpha > \alpha_n := 1 + max\{\frac{2}{n}, \frac{4(p-2)}{n(p-1)}\}$ such that $f(\lambda u, \lambda u_t, \lambda \nabla u) = \lambda^\alpha f(u, u_t, \nabla u)$, $\forall \lambda > 0$.

We note that the above assumptions are similar to that in [1], which corresponds to the case $p = 2$. And we will use the operator $|\nabla|$ (which is defined by the Fourier transform) to measure regularity. If we let

$$\sigma_p(r,n) = (\tfrac{n}{2} + r)(p-1) + r \tag{1.2}$$

denote the index of regularity-loss, then we can state our main theorem:

Theorem 1.1 (existence and decay estimates)**.** *Let $s \geq 0, p \geq 1$ be real numbers, $s > max\{\frac{p}{2} + 1, p - \frac{1}{2}\}$ for $n = 1$ and $s \geq \frac{np}{2} + 1$ for $n \geq 2$. Assume that $u_0 \in H^{s+max\{1,p-1\}}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and $u_1 \in H^s(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, and put*

$$E_0 := \|u_0\|_{H^{s+max\{1,p-1\}}} + \|u_1\|_{H^s} + \|(u_0, u_1)\|_{L^1}.$$

Then there exists a unique solution $u(x,t)$ to the problem (1.1)*, which satisfies*

$$u \in C^0([0,\infty); H^{s+max\{1,p-1\}}(\mathbb{R}^n)) \cap C^1([0,\infty); H^s(\mathbb{R}^n)),$$

and the following decay estimates:

$$\||\nabla|^r u(t)\|_{H^{s+max\{1,p-1\}-\sigma_p(r,n)}} \leq CE_0(1+t)^{-\frac{n}{4}-\frac{r}{2}}, \tag{1.3}$$

for real number $r \geq 0$ satisfying $\sigma_p(r,n) \leq s + max\{1, p-1\}$, and

$$\||\nabla|^r u_t(t)\|_{H^{s-\sigma_p(r,n)}} \leq CE_0(1+t)^{-\frac{n}{4}-\frac{r}{2}}, \tag{1.4}$$

for real number $r \geq 0$ satisfying $\sigma_p(r,n) \leq s$.

Remark 1. *The case $p = 2$ corresponds essentially to the result in [2].*

For the plate type equations, there are many results in the literature. In [4], da Luz and Charão studied a semilinear damped dissipative plate equation:

$$u_{tt} - \Delta u_{tt} + \Delta^2 u + u_t = f(u). \tag{1.5}$$

They proved the global existence of solutions and a polynomial decay of the energy by exploiting an energy estimate method. However their result was restricted to the dimension $1 \le n \le 5$. This restriction on the space dimension was removed by Sugitani-Kawashima [5] by the fundamental method of energy estimates in the Fourier (or frequency) space and some sharp decay estimates. For the case of memory dissipative plate equations, Liu-Kawashima [3] studied the following semilinear plate equation with memory term

$$u_{tt} + \Delta^2 u + u + g * \Delta u = f(u),$$

and obtained the global existence and decay estimates of solutions by employing the energy method in the Fourier space. In [1], Liu studied the following problem with memory and rotational term

$$u_{tt} - \Delta u_{tt} + \Delta^2 u + u + g * \Delta u = f(u, u_t, \nabla u),$$

and proved similar results as in [3]. The results in these papers [1, 3] and the general dissipative plate equation [5, 6, 7] show that they are of regularity-loss property. The decay structure of the regularity-loss type in [1, 3] is characterized by a function in the frequency space

$$\rho(\xi) = \frac{|\xi|^2}{1 + |\xi|^4}.$$

A similar decay structure of the regularity-loss type was also observed for the dissipative Timoshenko system ([8]) and a hyperbolic-elliptic system related to a radiating gas ([9]). For more studies on various aspects of dissipation of plate equations, we refer to [10, 11, 12, 13]. Also, as for the study of decay properties for hyperbolic systems of memory-type dissipation, we refer to [14, 15, 16].

The main purpose of this paper is to study the decay estimates and regularity-loss property of solutions to the initial value problem (1.1) in the spirit of [1, 3]. In [2], Mao-Liu studied the linear equation corresponding to (1.1) and obtained a result which shows that in the case of $p > 1$, the decay structure of the linear equation is of regularity-loss property and this property is characterized by the following function in the frequency space

$$\rho_p(\xi) = \frac{|\xi|^2}{1 + |\xi|^{2p}}$$

while in the case of $p = 1$ there is no regularity-loss. Our goal is to check whether this property is stable under the semilinear perturbation. By a similar argument as in [1], we proved this stability under our assumptions. We note since our result is in the frame of fractional order derivative and fractional Sobolev spaces, more subtle and delicate estimates must be needed, and it will be done in Section 2.

Before closing this section, we give some notations to be used below. Let $\mathcal{F}[f]$ denote the Fourier transform of f defined by

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx,$$

and we denote its inverse transform as $\mathcal{F}^{-1}$.

For $s \in \mathbb{R}$, we denote the Sobolev spaces by $H^s(\mathbb{R}^n)$, its norm is defined by

$$\|f\|_{H^s} = \|(1-\Delta)^{\frac{s}{2}} f\|_{L^2(\mathbb{R}^n_x)} \cong \|\langle\xi\rangle^s \hat{f}\|_{L^2(\mathbb{R}^n_\xi)},$$

here $\langle\xi\rangle = (1 + |\xi|^2)^{\frac{1}{2}}$ denotes the Japanese bracket.

Also, $C^k(I; H^s(\mathbb{R}^n))$ denotes the space of k-times continuously differentiable functions on the interval I with values in the Sobolev space $H^s = H^s(\mathbb{R}^n)$.

Finally, in this paper, we denote various constants by the same symbol C or c, which may change line to line.

2 Proof of the Main Theorem

In this section, we prove the global existence and decay estimates for solution to the problem (1.1) by employing the contraction mapping theorem. First we recall some properties for the fundamental solution $G(x,t)$ and $H(x,t)$, which satisfy the following equations:

$$\begin{cases} (1-\Delta)G_{tt} + (1+(-\Delta)^p)G + g * \Delta G = 0, \\ G(x,0) = \delta(x), \quad G_t(x,0) = 0, \end{cases}$$

and

$$\begin{cases} (1-\Delta)H_{tt} + (1+(-\Delta)^p)H + g * \Delta H = 0, \\ H(x,0) = 0, \quad H_t(x,0) = \delta(x), \end{cases}$$

respectively.

Lemma 2.1 (see [2]). *Let $r, \mu, \nu, s \geq 0, p \geq 1$ be real numbers , $\varphi \in H^{s+max\{1,p-1\}}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, $\psi \in H^s(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, $1 \leq q \leq 2$, then the following estimates hold:*

Case I ($p > 1$) :

1) $\||\nabla|^r G(t) * \varphi\|_{H^\mu} \leq C(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{r}{2}}\|\varphi\|_{L^q} + C(1+t)^{-\frac{\nu}{2(p-1)}}\|\varphi\|_{H^{r+\mu+\nu}}$,
for $r \geq 0,\ \mu \geq 0,\ \nu \geq 0,\ r+\mu+\nu \leq s + max\{1, p-1\}$.

2) $\||\nabla|^r G_t(t) * \varphi\|_{H^\mu} \leq C(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{r}{2}}\|\varphi\|_{L^q} + C(1+t)^{-\frac{\nu}{2(p-1)}}\|\varphi\|_{H^{r+\mu+\nu+max\{1,p-1\}}}$,
for $r \geq 0,\ \mu \geq 0,\ \nu \geq 0,\ r+\mu+\nu \leq s$.

3) $\||\nabla|^r H(t) * \psi\|_{H^\mu} \leq C(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{r}{2}}\|\psi\|_{L^q} + C(1+t)^{-\frac{\nu}{2(p-1)}}\|\psi\|_{H^{r+\mu+\nu-max\{1,p-1\}}}$,
for $r \geq 0,\ \mu \geq 0,\ \nu \geq 0,\ r+\mu+\nu \leq s + max\{1, p-1\}$.

4) $\||\nabla|^r H_t(t) * \psi\|_{H^\mu} \leq C(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{r}{2}}\|\psi\|_{L^q} + C(1+t)^{-\frac{\nu}{2(p-1)}}\|\psi\|_{H^{r+\mu+\nu}}$,
for $r \geq 0,\ \mu \geq 0,\ \nu \geq 0,\ r+\mu+\nu \leq s$.

Case II ($p = 1$) :

1)′ $\||\nabla|^r G(t) * \varphi\|_{H^\mu} \leq C(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{r}{2}}\|\varphi\|_{L^q} + Ce^{-ct}\|\varphi\|_{H^{r+\mu}}$,
for $0 \leq r+\mu \leq s$.

2)′ $\||\nabla|^r G_t(t) * \varphi\|_{H^\mu} \leq C(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{r}{2}}\|\varphi\|_{L^q} + Ce^{-ct}\|\varphi\|_{H^{r+\mu}}$,
for $0 \leq r+\mu \leq s$.

3)′ $\||\nabla|^r H(t) * \varphi\|_{H^\mu} \leq C(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{r}{2}}\|\varphi\|_{L^q} + Ce^{-ct}\|\varphi\|_{H^{r+\mu}}$,
for $0 \leq r+\mu \leq s$.

4)′ $\||\nabla|^r H_t(t) * \varphi\|_{H^\mu} \leq C(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{r}{2}}\|\varphi\|_{L^q} + Ce^{-ct}\|\varphi\|_{H^{r+\mu}}$,
for $0 \leq r+\mu \leq s$.

Then in terms of $G(x,t)$ and $H(x,t)$, the solution $u(t,x)$ to the equation (1.1) can be formally written as

$$u(t) = G(t) * u_0 + H(t) * u_1 + \int_0^t H(t-\tau) * (1-\Delta)^{-1} f(u, u_t, \nabla u)(\tau)d\tau. \tag{2.1}$$

We also need the following lemma which can be proved by inductive argument combined with the Littlewood-Paley theory:

Lemma 2.2. *Assume that $\alpha \geq 1$ and $\beta \geq 1$ are integers, then the following estimates hold:*

(1). $\|\partial_x^m(u^\alpha v^\beta)\|_{L^1} \leq C\|u\|_{L^\infty}^{\alpha-1}\|v\|_{L^\infty}^{\beta-1}(\|u\|_{L^2}\|\partial_x^m v\|_{L^2} + \|v\|_{L^2}\|\partial_x^m u\|_{L^2}),\ \forall m \in \mathbb{Z}^+$.

(2). $\||\nabla|^r(u^\alpha v^\beta)\|_{L^2} \leq C\|u\|_{L^\infty}^{\alpha-1}\|v\|_{L^\infty}^{\beta-1}(\|u\|_{L^\infty}\||\nabla|^r v\|_{L^2} + \|v\|_{L^\infty}\||\nabla|^r u\|_{L^2}), \forall r \in \mathbb{R}^+$.

Finally we recall the decay estimates for solutions to the linear problem (i.e., with $f(u, u_t, \nabla u) = 0$ in (1.1)) in [2]:

$$\begin{cases} u_{tt} - \Delta u_{tt} + (-\Delta)^p u + u + g * \Delta u = 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x). \end{cases} \tag{2.2}$$

Lemma 2.3 (see [2]). *Let $s \geq 0$ be a real number. Assume that $u_0 \in H^{s+max\{1,p-1\}}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and $u_1 \in H^s(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, and put*

$$E_0 := \|u_0\|_{H^{s+max\{1,p-1\}}} + \|u_1\|_{H^s} + \|(u_0, u_1)\|_{L^1}.$$

Let u be the solution to (2.2), then u satisfies the following decay estimates:

(1). If $s \geq \frac{n}{2}(p-1) - max\{1, p-1\}$, then

$$\||\nabla|^r u(t)\|_{H^{s+max\{1,p-1\}-\sigma_p(r,n)}} \leq CE_0(1+t)^{-\frac{n}{4}-\frac{r}{2}},$$

for $r \geq 0$ and $\sigma_p(r,n) \leq s + max\{1, p-1\}$.

(2). If $s \geq \frac{n}{2}(p-1)$, then

$$\||\nabla|^r u_t(t)\|_{H^{s-\sigma_p(r,n)}} \leq CE_0(1+t)^{-\frac{n}{4}-\frac{r}{2}},$$

for $r \geq 0$ and $\sigma_p(r,n) \leq s$, here $\sigma_p(r,n)$ are defined in (1.2).

Remark 2. *We state the results in the previous lemma in somewhat different form, and it can be directly verified by checking the argument in [2].*

Now in order to prove theorem 1.1, we define

$$X := \{u \in C^0([0,\infty); H^{s+max\{1,p-1\}}(\mathbb{R}^n)) \cap C^1([0,\infty); H^s(\mathbb{R}^n)); \|u\|_X < \infty\},$$

here

$$\begin{aligned} \|u\|_X := & \sup_{t\geq 0} \|u(t)\|_{H^{s+max\{1,p-1\}}} + \sup_{t\geq 0} \|u_t(t)\|_{H^s} \\ & + \sup_{\{r;\sigma_p(r,n)\leq s+max\{1,p-1\}\}} \sup_{t\geq 0}(1+t)^{\frac{n}{4}+\frac{r}{2}} \||\nabla|^r u(t)\|_{H^{s+max\{1,p-1\}-\sigma_p(r,n)}} \\ & + \sup_{\{r;\sigma_p(r,n)\leq s\}} \sup_{t\geq 0}(1+t)^{\frac{n}{4}+\frac{r}{2}} \||\nabla|^r u_t(t)\|_{H^{s-\sigma_p(r,n)}}. \end{aligned}$$

Denote

$$\begin{aligned} & U := (u, u_t, \nabla u), \\ & B_R := \{u \in X; \|u\|_X \leq R\}, \quad \forall R > 0, \\ & \phi[u](t) := G(t) * u_0 + H(t) * u_1 + \int_0^t H(t-\tau) * (1-\Delta)^{-1} f(U)(\tau) d\tau, \\ & \phi_0(t) := G(t) * u_0 + H(t) * u_1. \end{aligned}$$

In the following we will prove that $u \to \phi[u]$ is a contraction mapping on B_R for some small $R > 0$. First we give two propositions which will be frequently used in the subsequent computation.

Proposition 1. *If $u \in X$, then the following estimate holds:*

$$\|U(t)\|_{L^\infty} \leq C\|u\|_X(1+t)^{-d_n}, \tag{2.3}$$

here

$$d_n = \begin{cases} \frac{1}{2}, & n = 1, \\ \frac{n}{4}, & n \geq 2. \end{cases}$$

Proof. By the Gagliardo-Nirenberg inequality, we have

$$\|U(t)\|_{L^\infty} \leq C\|U(t)\|_{L^2}^{1-\theta}\||\nabla|^{s_0}U(t)\|_{L^2}^{\theta},$$

here $s_0 = \frac{n}{2} + \epsilon_0$, with $\epsilon_0 \in \big(0, (s - p + \frac{1}{2})/p\big]$ fixed, and $\theta = \frac{n}{2s_0}$.

(1). When $n = 1$, since $s \geq \frac{p}{2} + 1$, we have $s + max\{1, p-1\} - \sigma_p(0, 1) \geq 1$, thus $\|U(t)\|_{L^2} \leq (1+t)^{-\frac{1}{4}}\|u\|_X$ by the definition of $\|u\|_X$. Similarly, since $s > p - \frac{1}{2}$, which implies $s - \sigma_p(s_0, 1) \geq 0$, we have $\||\nabla|^{s_0}U(t)\|_{L^2} \leq (1+t)^{-\frac{1}{4}-\frac{s_0}{2}}\|u\|_X$. Then $d_n = (1-\theta)\frac{1}{4} + \theta(\frac{1}{4} + \frac{s_0}{2}) = \frac{1}{2}$, it yields (2.3) with $n = 1$.

(2). When $n \geq 2$, since $s \geq \frac{np}{2} + 1$, we have $s - \sigma_p(0, n) \geq s_0$, thus $\|U(t)\|_{L^2} \leq (1+t)^{-\frac{n}{4}}\|u\|_X$ and $\||\nabla|^{s_0}U(t)\|_{L^2} \leq (1+t)^{-\frac{n}{4}}\|u\|_X$. Then $d_n = (1-\theta)\frac{n}{4} + \theta\frac{n}{4} = \frac{n}{4}$, it yields (2.3) with $n \geq 2$. □

Proposition 2. *Let $a \geq 0$ and $b \geq 0$ be real numbers. If $a + b \geq 1$, then there exists $C > 0$ (independent of $t > 0$) such that the following estimate holds,*

$$\int_0^t (1+t-\tau)^{-a}(1+\tau)^{-b}d\tau \leq C.$$

Proof. Directly computation! □

Proof of Theorem 1.1. We denote $V := (v, v_t, \nabla v), W := (w, w_t, \nabla w)$, then

$$\phi[v](t) - \phi[w](t) = \int_0^t H(t-\tau) * (1-\Delta)^{-1}(f(V) - f(W))(\tau)d\tau.$$

Case I ($p > 1$): We split the proof into four steps.

Step 1: By applying Lemma 2.1 3) with $q = 1$ and $\mu = s + max\{1, p-1\}$, we have that

$$\begin{aligned} \|(\phi[v] - \phi[w])(t)\|_{H^{s+max\{1,p-1\}}} &= \|\int_0^t H(t-\tau) * (f(V) - f(W))(\tau)d\tau\|_{H^{s+max\{1,p-1\}-2}} \\ &\leq C\int_0^t (1+t-\tau)^{-\frac{n}{4}}\|f(V) - f(W)\|_{L^1}d\tau \\ &\quad + C\int_0^t (1+t-\tau)^{-\frac{\nu}{2(p-1)}}\|f(V) - f(W)\|_{H^{s-2+\nu}}d\tau \\ &= I_1 + I_2. \end{aligned} \tag{2.4}$$

By using Lemma 2.2, we get that

$$\|(f(V) - f(W))(\tau)\|_{L^1} \leq C\|(V, W)(\tau)\|_{L^\infty}^{\alpha-2}\|(V, W)(\tau)\|_{L^2}\|(V - W)(\tau)\|_{L^2}.$$

In view of (2.3), we have that

$$\begin{aligned} \|(f(V) - f(W))(\tau)\|_{L^1} &\leq C\|(v, w)\|_X^{\alpha-1}\|v - w\|_X(1+\tau)^{-d_n(\alpha-2)-\frac{n}{2}} \\ &\leq C\|(v, w)\|_X^{\alpha-1}\|v - w\|_X \begin{cases} (1+\tau)^{-\frac{\alpha-1}{2}}, & n = 1, \\ (1+\tau)^{-\frac{n\alpha}{4}}, & n \geq 2. \end{cases} \end{aligned} \tag{2.5}$$

Then by Assumption [B] and Proposition 2, we have

$$I_1 \leq C\|(v, w)\|_X^{\alpha-1}\|v - w\|_X. \tag{2.6}$$

Taking $\nu = 2$, by using Lemma 2.2, it holds that

$$\begin{aligned}\|f(V)-f(W)\|_{H^s} \leq \quad & C\|(v,w)\|_{L^\infty}^{\alpha-2}(\|(v,w)\|_{L^\infty}\|v-w\|_{H^s} \\ & +\|(v,w)\|_{H^s}\|v-w\|_{L^\infty}).\end{aligned} \tag{2.7}$$

It yields that

$$\|f(V)-f(W)\|_{H^s} \leq C\|(v,w)\|_X^{\alpha-1}\|v-w\|_X(1+\tau)^{-d_n(\alpha-1)}. \tag{2.8}$$

Then by Assumption [B] and Proposition 2, we have

$$I_2 \leq C\|(v,w)\|_X^{\alpha-1}\|v-w\|_X.$$

Put the estimates for I_1 and I_2 in (2.4), then we obtain

$$\|(\phi[v]-\phi[w])(t)\|_{H^{s+max\{1,p-1\}}} \leq C\|(v,w)\|_X^{\alpha-1}\|v-w\|_X. \tag{2.9}$$

Step 2: By applying Lemma 2.1 4) with $q = 1$, $\mu = s-2$ and $\nu = 2$, we have that

$$\begin{aligned}\|\partial_t(\phi[v]-\phi[w])(t)\|_{H^s} \quad & = \|\int_0^t H_t(t-\tau)*(f(V)-f(W))(\tau)d\tau\|_{H^{s-2}} \\ & \leq C\int_0^t(1+t-\tau)^{-\frac{n}{4}}\|f(V)-f(W)\|_{L^1}d\tau \\ & \quad +C\int_0^t(1+t-\tau)^{-\frac{1}{p-1}}\|f(V)-f(W)\|_{H^s}d\tau \\ & = I_3+I_4.\end{aligned} \tag{2.10}$$

We can estimate I_3 and I_4 similarly as for I_1 and I_2 in step 1, then we have

$$\|\partial_t(\phi[v]-\phi[w])(t)\|_{H^s} \leq C\|(v,w)\|_X^{\alpha-1}\|v-w\|_X. \tag{2.11}$$

Step 3: Let $r \geq 0$ be a real number satisfying $\sigma_p(r,n) \leq s+max\{1,p-1\}$, and $\mu = s+max\{1,p-1\}-\sigma_p(r,n)$, then we have

$$\begin{aligned}& \||\nabla|^r(\phi[v]-\phi[w])(t)\|_{H^{s+max\{1,p-1\}-\sigma_p(r,n)}} \\ & \leq (\int_0^{\frac{t}{2}}+\int_{\frac{t}{2}}^t)\||\nabla|^{r-[r]}H(t-\tau)*|\nabla|^{[r]}(f(V)-f(W))\|_{H^{s+max\{1,p-1\}-\sigma_p(r,n)-2}}d\tau \\ & = I_5+I_6.\end{aligned} \tag{2.12}$$

By virtue of Lemma 2.1 3), we have

$$\begin{aligned}I_5 \leq & C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{4}-\frac{r-[r]}{2}}\|\partial_x^{[r]}(f(V)-f(W))\|_{L^1}d\tau \\ & +C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{\nu}{2(p-1)}}\|\partial_x^{[r]}(f(V)-f(W))\|_{H^{r-[r]+s-\sigma_p(r,n)-2+\nu}}d\tau \\ = & I_{5a}+I_{5b}.\end{aligned} \tag{2.13}$$

In view of (2.5) and by a similar argument to (2.6), we have that

$$\begin{aligned}\|\partial_x^{[r]}(f(V)-f(W))\|_{L^1} \leq & C\|(V,W)(\tau)\|_{L^\infty}^{\alpha-2}(\|(V,W)(\tau)\|_{L^2}\|\partial_x^{[r]}(V-W)(\tau)\|_{L^2} \\ & +\|\partial_x^{[r]}(V,W)(\tau)\|_{L^2}\|(V-W)(\tau)\|_{L^2}) \\ \leq & C\|(v,w)\|_X^{\alpha-1}\|v-w\|_X(1+\tau)^{-d_n(\alpha-2)-\frac{n}{2}-\frac{[r]}{2}}.\end{aligned} \tag{2.14}$$

Thus

$$I_{5a} \le C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{4}-\frac{r-[r]}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X(1+\tau)^{-d_n(\alpha-2)-\frac{n}{2}-\frac{[r]}{2}}d\tau$$
$$\le C(1+t)^{-\frac{n}{4}-\frac{r}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X.$$

If $r+\mu-2+\nu-max\{1,p-1\}<s+max\{1,p-1\}$, taking $\nu=\sigma_p(r,n)-r+max\{1,p-1\}$, then $\frac{\nu}{2(p-1)}\ge\frac{n}{4}+\frac{r}{2}+\frac{1}{2}$, by virtue of (2.7) we have

$$\begin{aligned}\|\partial_x^{[r]}(f(V)-f(W))\|_{H^{s+max\{1,p-1\}-[r]-2}} \\ \le\ C\|(v,w)\|_{L^\infty}^{\alpha-2}(\|(v,w)\|_{L^\infty}\|\partial_x^{[r]}(v-w)\|_{H^{s+max\{1,p-1\}-[r]-2} \\ +\|\partial_x^{[r]}(v,w)\|_{H^{s+max\{1,p-1\}-[r]-2}}\|v-w\|_{L^\infty}).\end{aligned} \tag{2.15}$$

It yields that

$$\|\partial_x^{[r]}(f(V)-f(W))\|_{H^{s+max\{1,p-1\}-[r]-2}}\le C\|(v,w)\|_X^{\alpha-1}\|v-w\|_X(1+\tau)^{-d_n(\alpha-1)}.$$

Since $\alpha>1$ by Assumption [B], we have

$$I_{5b}\le C(1+t)^{-\frac{n}{4}-\frac{r}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X.$$

Put the estimates for I_{5a} and I_{5b} in (2.13), we obtain that

$$I_5\le C(1+t)^{-\frac{n}{4}-\frac{r}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X. \tag{2.16}$$

In view of Lemma 2.1 3) with $\nu=p-1$, we have that

$$\begin{aligned}I_6\ &\le C\textstyle\int_{\frac{t}{2}}^t(1+t-\tau)^{-\frac{n}{4}-\frac{r-[r]}{2}}\|\partial_x^{[r]}(f(V)-f(W))\|_{L^1}d\tau \\ &\quad+C\textstyle\int_{\frac{t}{2}}^t(1+t-\tau)^{-\frac{\nu}{2(p-1)}}\|\partial_x^{[r]}(f(V)-f(W))\|_{H^{r-[r]+s-2+p-1}}d\tau \\ &=I_{6a}+I_{6b}.\end{aligned} \tag{2.17}$$

Since

$$\begin{aligned}\|\partial_x^{[r]}(f(V)-f(W))\|_{L^1}&\le C\|(V,W)(\tau)\|_{L^\infty}^{\alpha-2}(\|(V,W)(\tau)\|_{L^2}\|\partial_x^{[r]}(V-W)(\tau)\|_{L^2} \\ &\quad+\|\partial_x^{[r]}(V,W)(\tau)\|_{L^2}\|(V-W)(\tau)\|_{L^2}) \\ &\le C\|(v,w)\|_X^{\alpha-1}\|v-w\|_X(1+\tau)^{-d_n(\alpha-2)-\frac{n}{2}-\frac{[r]}{2}}.\end{aligned} \tag{2.18}$$

We have that

$$I_{6a}\le C\int_{\frac{t}{2}}^t(1+t-\tau)^{-\frac{n}{4}-\frac{r-[r]}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X(1+\tau)^{-d_n(\alpha-2)-\frac{n}{2}-\frac{[r]}{2}}d\tau$$
$$\le C(1+t)^{-\frac{n}{4}-\frac{r}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X.$$

By virtue of (2.15), we obtain

$$\|\partial_x^{[r]}(f(V)-f(W))\|_{H^{r-[r]+s-2+p-1}}\le C\|(v,w)\|_X^{\alpha-1}\|v-w\|_X(1+\tau)^{-d_n(\alpha-1)-\frac{n}{4}-\frac{[r]}{2}}.$$

By virtue of Assumption [B], we have

$$I_{6b}\le C(1+t)^{-\frac{n}{4}-\frac{r}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X.$$

Put the estimates for I_{6a} and I_{6b} in (2.17), we obtain that

$$I_6\le C(1+t)^{-\frac{n}{4}-\frac{r}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X. \tag{2.19}$$

Combining the estimates (2.12) (2.16), and (2.19) we obtain that

$$\||\nabla|^r(\phi[v]-\phi[w])(t)\|_{H^{s+max\{1,p-1\}-\sigma_p(r,n)}} \le C(1+t)^{-\frac{n}{4}-\frac{r}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X.$$

It yields that

$$\sup_{t\ge 0}(1+t)^{\frac{n}{4}+\frac{r}{2}}\||\nabla|^r(\phi[v]-\phi[w])(t)\|_{H^{s+max\{1,p-1\}-\sigma_p(r,n)}} \le C\|(v,w)\|_X^{\alpha-1}\|v-w\|_X. \quad (2.20)$$

Step 4: Assume that $r\ge 0$ be a real number with $\sigma_p(r,n)\le s$ and $\mu=s-\sigma_p(r,n)$, then we have

$$\begin{aligned}&\||\nabla|^r\partial_t(\phi[v]-\phi[w])(t)\|_{H^{s-\sigma_p(r,n)}}\\ &\le (\int_0^{\frac{t}{2}}+\int_{\frac{t}{2}}^t)\||\nabla|^{r-[r]}H_t(t-\tau)*|\nabla|^{[r]}(f(V)-f(W))\|_{H^{s-\sigma_p(r,n)-2}}d\tau\\ &= I_7+I_8. \end{aligned} \quad (2.21)$$

By virtue of Lemma 2.1 4), we have

$$\begin{aligned}I_7 &\le C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{4}-\frac{r-[r]}{2}}\|\partial_x^{[r]}(f(V)-f(W))\|_{L^1}d\tau\\ &\quad + C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{\nu}{2(p-1)}}\|\partial_x^{[r]}(f(V)-f(W))\|_{H^{r-[r]+s-\sigma_p(r,n)-2+\nu}}d\tau\\ &= I_{7a}+I_{7b}.\end{aligned} \quad (2.22)$$

Similar to the proof of (2.6), we obtain

$$I_{7a}\le C(1+t)^{-\frac{n}{4}-\frac{r}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X.$$

If $r+\mu-2+\nu\le s$, by virtue of (2.15), with $r-[r]+\mu-2+\nu-max\{1,p-1\}$ replaced by $r+\mu-2+\nu$, taking $\nu=\sigma_p(r,n)-r$, then $\frac{\nu}{2(p-1)}=\frac{n}{4}+\frac{r}{2}$ and we have

$$\|\partial_x^{[r]}(f(V)-f(W))\|_{H^{s-[r]-2}}\le C\|(v,w)\|_X^{\alpha-1}\|v-w\|_X(1+\tau)^{-d_n(\alpha-1)-\frac{n}{4}-\frac{r}{2}}.$$

Thus we have that

$$I_{7b}\le C(1+t)^{-\frac{n}{4}-\frac{r}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X.$$

Put the estimates for I_{7a} and I_{7b} in (2.22), we obtain that

$$I_7\le C(1+t)^{-\frac{n}{4}-\frac{r}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X. \quad (2.23)$$

By virtue of Lemma 2.1 4) with $\nu=2$, we have

$$\begin{aligned}I_8 &\le C\int_{\frac{t}{2}}^t(1+t-\tau)^{-\frac{n}{4}-\frac{r-[r]}{2}}\|\partial_x^{[r]}(f(V)-f(W))\|_{L^1}d\tau\\ &\quad +C\int_{\frac{t}{2}}^t(1+t-\tau)^{-\frac{\nu}{2(p-1)}}\|\partial_x^{[r]}(f(V)-f(W))\|_{H^{r-[r]+s-\sigma_p(r,n)}}d\tau\\ &= I_{8a}+I_{8b}.\end{aligned} \quad (2.24)$$

Since

$$\begin{aligned}\|\partial_x^{[r]}(f(V)-f(W))\|_{L^1}\le C\|(V,W)(\tau)\|_{L^\infty}^{\alpha-2}(&\|(V,W)(\tau)\|_{L^2}\|\partial_x^{[r]}(V-W)(\tau)\|_{L^2}\\ &+\|\partial_x^{[r]}(V,W)(\tau)\|_{L^2}\|(V-W)(\tau)\|_{L^2}).\end{aligned}$$

thus

$$\|\partial_x^{[r]}(f(V)-f(W))\|_{L^1}\le C\|(v,w)\|_X^{\alpha-1}\|v-w\|_X(1+\tau)^{-d_n(\alpha-2)-\frac{n}{2}-\frac{[r]}{2}}.$$

It yields that

$$I_{8a} \le C(1+t)^{-\frac{n}{4}-\frac{r}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X.$$

Similar to the proof of I_{7b}, we obtain

$$I_{8b} \le C(1+t)^{-\frac{n}{4}-\frac{r}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X.$$

Put the estimates for I_{8a} and I_{8b} in (2.24), we obtain that

$$I_8 \le C(1+t)^{-\frac{n}{4}-\frac{r}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X. \tag{2.25}$$

Combining (2.21), (2.23) and (2.25) we obtain that

$$\||\nabla|^r\partial_t(\phi[v]-\phi[w])(t)\|_{H^{s-\sigma_p(r,n)}} \le C(1+t)^{-\frac{n}{4}-\frac{r}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X.$$

It yields that

$$\sup_{t\ge 0}(1+t)^{\frac{n}{4}+\frac{r}{2}}\||\nabla|^r\partial_t(\phi[v]-\phi[w])(t)\|_{H^{s-\sigma_p(r,n)}} \le C\|(v,w)\|_X^{\alpha-1}\|v-w\|_X. \tag{2.26}$$

Case II ($p=1$): We split the proof into four steps.

Step $1'$**:** In view of Lemma 2.1 3)$'$, with $q=1, \mu=s+1$, we have that

$$\begin{aligned}\|(\phi[v]-\phi[w])(t)\|_{H^{s+1}} &= \|\int_0^t H(t-\tau)*(f(V)-f(W))(\tau)d\tau\|_{H^{s+1-2}} \\ &\le C\int_0^t(1+t-\tau)^{-\frac{n}{4}}\|f(V)-f(W)\|_{L^1}d\tau \\ &\quad +C\int_0^t e^{-c(t-\tau)}\|f(V)-f(W)\|_{H^{s+1-2}}d\tau \\ &= J_1+J_2.\end{aligned} \tag{2.27}$$

By a similar proof to (2.5) and (2.7), we have that

$$J_1 \le C\|(v,w)\|_X^{\alpha-1}\|v-w\|_X, \qquad J_2 \le C\|(v,w)\|_X^{\alpha-1}\|v-w\|_X.$$

Put the estimates for J_1 and J_2 in (2.27), we obtain

$$\|(\phi[v]-\phi[w])(t)\|_{H^{s+1}} \le C\|(v,w)\|_X^{\alpha-1}\|v-w\|_X, \tag{2.28}$$

for $p=1$.

Step $2'$**:** In view of Lemma 2.1 4)$'$ with $q=1$ and $\mu=s$, we have that

$$\begin{aligned}\|\partial_t(\phi[v]-\phi[w])(t)\|_{H^s} &= \|\int_0^t H_t(t-\tau)*(f(V)-f(W))(\tau)d\tau\|_{H^{s-2}} \\ &\le C\int_0^t(1+t-\tau)^{-\frac{n}{4}}\|f(V)-f(W)\|_{L^1}d\tau \\ &\quad +C\int_0^t e^{-c(t-\tau)}\|f(V)-f(W)\|_{H^{s-2}}d\tau \\ &= J_3+J_4.\end{aligned} \tag{2.29}$$

We can estimate J_3 and J_4 similarly as for J_1 and J_2 in step $1'$, then we have

$$\|\partial_t(\phi[v]-\phi[w])(t)\|_{H^s} \le C\|(v,w)\|_X^{\alpha-1}\|v-w\|_X, \tag{2.30}$$

for $p=1$.

Step $3'$**:** Assume that $r \geq 0$ be a real number with $\sigma_p(r,n) \leq s+1$ and $\mu = s+1-\sigma_p(r,n)$, then we have

$$\begin{aligned}
&\||\nabla|^r(\phi[v]-\phi[w])(t)\|_{H^{s+1-\sigma_p(r,n)}} \\
&\leq (\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t)\||\nabla|^{r-[r]}H(t-\tau) * |\nabla|^{[r]}(f(V)-f(W))\|_{H^{s+1-\sigma_p(r,n)-2}} d\tau \\
&= J_5 + J_6. \qquad (2.31)
\end{aligned}$$

In view of Lemma 2.1 $3)'$, we have that

$$\begin{aligned}
J_5 \quad &\leq C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{4}-\frac{r-[r]}{2}}\|\partial_x^{[r]}(f(V)-f(W))\|_{L^1}d\tau \\
&+C\int_0^{\frac{t}{2}} e^{-c(t-\tau)}\|\partial_x^{[r]}(f(V)-f(W))\|_{H^{r-[r]+s+1-\sigma_p(r,n)-2}}d\tau \\
&= J_{5a} + J_{5b}.
\end{aligned} \qquad (2.32)$$

Similar to the proof of I_{5a} and I_{5b}, we obtain

$$J_{5a} \leq C(1+t)^{-\frac{n}{4}-\frac{r}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X,$$

and

$$J_{5b} \leq C(1+t)^{-\frac{n}{4}-\frac{r}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X.$$

Thus

$$J_5 \leq C(1+t)^{-\frac{n}{4}-\frac{r}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X. \qquad (2.33)$$

In view of Lemma 2.1 $3)'$, we have that

$$\begin{aligned}
J_6 \quad &\leq C\int_{\frac{t}{2}}^t(1+t-\tau)^{-\frac{n}{4}-\frac{r-[r]}{2}}\|\partial_x^{[r]}(f(V)-f(W))\|_{L^1}d\tau \\
&+C\int_{\frac{t}{2}}^t e^{-c(t-\tau)}\|\partial_x^{[r]}(f(V)-f(W))\|_{H^{r+s+1-\sigma_p(r,n)-2}}d\tau \\
&= J_{6a} + J_{6b}.
\end{aligned} \qquad (2.34)$$

Similar to the proof of I_{6a} and I_{6b}, we obtain

$$J_{6a} \leq C(1+t)^{-\frac{n}{4}-\frac{r}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X,$$

and

$$J_{6b} \leq C(1+t)^{-\frac{n}{4}-\frac{r}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X.$$

Thus

$$J_6 \leq C(1+t)^{-\frac{n}{4}-\frac{r}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X. \qquad (2.35)$$

Combining the estimates (2.31) (2.33), and (2.35) we obtain that

$$\||\nabla|^r(\phi[v]-\phi[w])(t)\|_{H^{s+1-\sigma_p(r,n)}} \leq C(1+t)^{-\frac{n}{4}-\frac{r}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X.$$

It yields that

$$\sup_{t\geq 0}(1+t)^{\frac{n}{4}+\frac{r}{2}}\||\nabla|^r(\phi[v]-\phi[w])(t)\|_{H^{s+1-\sigma_p(r,n)}} \leq C\|(v,w)\|_X^{\alpha-1}\|v-w\|_X. \qquad (2.36)$$

Step $4'$**:** Assume that $r \geq 0$ be a real number with $\sigma_p(r,n) \leq s$ and $\mu = s-\sigma_p(r,n)$, then we have

$$\begin{aligned}
&\||\nabla|^r\partial_t(\phi[v]-\phi[w])(t)\|_{H^{s-\sigma_p(r,n)}} \\
&\leq (\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t)\||\nabla|^{r-[r]}H_t(t-\tau) * |\nabla|^{[r]}(f(V)-f(W))\|_{H^{s-\sigma_p(r,n)-2}} d\tau \\
&= J_7 + J_8. \qquad (2.37)
\end{aligned}$$

In view of Lemma 2.1 4)′, we have that

$$\begin{aligned} J_7 &\le C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{4}-\frac{r-[r]}{2}}\|\partial_x^{[r]}(f(V)-f(W))\|_{L^1}d\tau \\ &+C\int_0^{\frac{t}{2}}e^{-c(t-\tau)}\|\partial_x^{[r]}(f(V)-f(W))\|_{H^{r-[r]+s-\sigma_p(r,n)}}d\tau \\ &= J_{7a}+J_{7b}. \end{aligned} \tag{2.38}$$

Similar to the proof of I_{7a} and I_{7b}, we obtain

$$J_{7a}\le C(1+t)^{-\frac{n}{4}-\frac{r}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X,$$

and

$$J_{7b}\le C(1+t)^{-\frac{n}{4}-\frac{r}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X.$$

Thus

$$J_7\le C(1+t)^{-\frac{n}{4}-\frac{r}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X. \tag{2.39}$$

By virtue of Lemma 2.1 4)′, we have

$$\begin{aligned} J_8 &\le C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{n}{4}-\frac{r-[r]}{2}}\|\partial_x^{[r]}(f(V)-f(W))\|_{L^1}d\tau \\ &+C\int_{\frac{t}{2}}^{t}e^{-c(t-\tau)}\|\partial_x^{[r]}(f(V)-f(W))\|_{H^{r-[r]+s-\sigma_p(r,n)}}d\tau \\ &= J_{8a}+J_{8b}. \end{aligned} \tag{2.40}$$

Similar to the proof of I_{8a} and I_{8b}, we obtain

$$J_{8a}\le C(1+t)^{-\frac{n}{4}-\frac{r}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X,$$

and

$$J_{8b}\le C(1+t)^{-\frac{n}{4}-\frac{r}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X.$$

Put the estimates for J_{8a} and J_{8b} in (2.40), we obtain that

$$J_8\le C(1+t)^{-\frac{n}{4}-\frac{r}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X. \tag{2.41}$$

Combining (2.37), (2.39) and (2.41) we obtain that

$$\||\nabla|^r\partial_t(\phi[v]-\phi[w])(t)\|_{H^{s-\sigma_p(r,n)}}\le C(1+t)^{-\frac{n}{4}-\frac{r}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X.$$

It yields that

$$\sup_{t\ge0}(1+t)^{\frac{n}{4}+\frac{r}{2}}\||\nabla|^r\partial_t(\phi[v]-\phi[w])(t)\|_{H^{s-\sigma_p(r,n)}}\le C\|(v,w)\|_X^{\alpha-1}\|v-w\|_X. \tag{2.42}$$

Step 5: Combining the estimates (2.9), (2.11), (2.20) and (2.26) for $p>1$; (2.28), (2.30), (2.36) and (2.42) for $p=1$, we obtain that

$$\|(\phi[v]-\phi[w])(t)\|_X\le C\|(v,w)\|_X^{\alpha-1}\|v-w\|_X.$$

So far we proved that $\|(\phi[v]-\phi[w])(t)\|_X\le C_1R^{\alpha-1}\|v-w\|_X$ if $v,w\in B_R$. If E_0 is suitably small such that $R<1$ and $C_1R\le\frac{1}{2}$, then we have that

$$\|(\phi[v]-\phi[w])(t)\|_X\le\frac{1}{2}\|v-w\|_X.$$

On the other hand, from Lemma 2.3 we know that $\|\phi_0\|_X\le C_2E_0$. Since $\phi[0](t)=\phi_0(t)$, by taking $R=2C_2E_0$, it yields that, for $v\in B_R$,

$$\|\phi[v]\|_X\le\|\phi_0\|_X+\frac{1}{2}\|v\|_X\le C_2E_0+\frac{1}{2}R=R.$$

Thus $v\to\phi[v]$ is a contraction mapping on B_R, and by the fixed point principle there exists a unique $u\in B_R$ satisfying $\phi[u]=u$, and it is the solution to the semilinear problem (1.1) satisfying the decay estimates (1.3) and (1.4). Thus we complete the proof of Theorem 1.1. □

Conclusion

In this paper, we studied the semilinear regularity-loss type equation with memory. By the time-weighted energy estimates and the contracting theorem, we proved the global existence and the decay estimate, as well as the regularity-loss estimates.

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Competing Interests

Authors have declared that no competing interests exist.

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