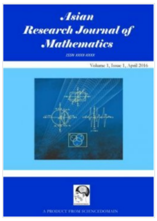

**Asian Research Journal of Mathematics**

**5(1): 1-11, 2017; Article no.ARJOM.34052**

*ISSN: 2456-477X*

# A Simple Model of Sediment Transport in the Nearshore Zone

**Evans F. Osaisai**[1*]

[1] *Department of Mathematics and Computer Sciences, Niger Delta University, PO Box 1693, Amassoma, Bayelsa State, Nigeria.*

***Author's contribution***

*The sole author designed, analyzed and interpreted and prepared the manuscript.*

***Article Information***

DOI: 10.9734/ARJOM/2017/34052

*Editor(s):*

(1) Gongxian Xu, Associate Professor, Department of Mathematics, Bohai University, Jinzhou, China.

*Reviewers:*

(1) Jagdish Prakash, University of Botswana, Botswana.

(2) Sumit Gandhi, Jaypee University of Engineering and Technology, Guna, India.

(3) Rajkumar V. Raikar, KLE Dr. M. S. Sheshgiri College of Engineering and Technology, India.

Complete Peer review History: http://www.sciencedomain.org/review-history/19865

***Received:** 11th May 2017*

***Accepted:** 12th June 2017*

***Published:** 4th July 2017*

***Original Research Article***

## Abstract

In this paper we examine a simple model of sediment transport, induced by the breaking waves in the surf zone. Essentially the bottom is allowed to move in response to the divergence of a sediment flux, in turn determined by the breaking waves. The effect of this extra term on the previous solutions for set-up, longshore currents and rip currents is then determined. It is found that the solutions for the mean flow are now unsteady on a slow timescale determined by a certain sediment transport parameter. There is a change in beach slope in the rip currents controlled by the sediment transport. The system of equations now forms a three-by-three nonlinear hyperbolic system of equations. These we solve approximately, using a simple wave solution based on the simple wave speed corresponding to the small sediment transport parameter. However, this solution will always breakdown after a long time, so we show that by adding another term proportional to the beach slope into the expression for the sediment flux, we can obtain a steady-state solution.

**Corresponding author: E-mail: fevansosaisai@gmail.com;*

*Keywords: Wave-current interactions; setup; set down; surf zone; shoaling zone; longshore currents.*

# 1 Introduction

The hydrodynamics of the nearshore zone constitutes an important area of studies. The interest is mainly driven by a combination of engineering, shipping and coastal interests. There has been much research on shoaling nonlinear waves, on how currents affect waves and how waves can drive currents. The basis for this subject was laid down by [1], [2], who analyzed the nonlinear interaction between short waves and long waves (or currents), and showed that variations in the energy of the short waves correspond to work done by the long waves against the *radiation stress* of the short waves. In the shoaling zone this radiation stress leads to what are now known as wave setup and wave setdown, surf beats, the generation of smaller waves by longer waves, and the steepening of waves on adverse currents, see [3], [4]. The divergence of the radiation stress was shown to generate an alongshore current by obliquely incident waves on a beach [5], [6]. During the shoaling of the waves there is a discontinuity in the wave energy in the mean vorticity equation of the waves. Thus there is sharp distinction in the behaviour of waves in the regimes due largely to forcing.

As wave groups propagate towards the shore, they enter shallower water and eventually break on beaches. The important process here is the wave breaking and dissipation of energy. The focusing of energy and the wave height variation across the group forces low frequency long waves that propagate with the group velocity [1]. These long waves may be amplified by continued forcing during the shoaling of the short wave group into shallower water [3], [7], [8] and [9]. In sufficiently shallow water, the short waves within the group may break at different depths leading to further long wave forcing by the varying breaker-line position [10], [11]. This means that the shoreward propagating waves may reflect at the shoreline and subsequently propagate offshore [12].

Wave breaking leads to a transfer of the incoming wave energy to a range of different scales of motions, and particularly to lower frequencies (see, for instance, [13]). Thus waves called surf beat [12], [14], may propagate in the cross-shore direction (called leaky waves). Waves may be trapped refractively as edge waves [15]. Wave breaking may occur for two reasons. Firstly due to natural variation in the wave direction and amplitude. These changes occur in space and time. Secondly wave may break due to topographic influences. When this is the case, as in the nearshore zone, the location and form of the wave breaking is influenced by the bottom depth profile.

Essentially we derived a model for the interaction between waves and currents. The aim is to provide analytical solution for waves in the nearshore zone on time scales longer than an individual wave. This is possible on long-time scales using the wave-averaging procedure often employed in the literature (see the textbook [16]). We describe solutions for rip currents, in the shoaling zone matched to the surf zone, for two different beach profiles.

The structure of the mathematical model is based on the Euler equations for an inviscid incompressible fluid. We then employ an averaging over the phase of the waves, exploiting the difference in time scales between the waves and the mean flow, which is our main interest. The nearshore zone is divided into regions, a shoaling zone where the wave field can be described by linear sinusoidal waves, and the surf zone, where the breaking waves are modelled empirically. The breakerline is fixed at $x = x_b$ but in general could vary.

In the shoaling zone wave field, we use an equation set consisting of a wave action equation, combined with the local dispersion relation and the wave kinematic equation for conservation of waves. The mean flow field is then obtained from a conservation of mass equation for the mean flow, and a momentum equation for the mean flow driven by the wave radiation stress tensor. In the surf zone, we use a standard empirical formula for the breaking wave field, together with the same mean flow equations.

# 2 Governing Equations and Fundamentals

With sediment transport, the kinematic bottom boundary condition becomes

$$w + h_t + \mathbf{u}.\nabla h = 0\,, \quad \text{at} \quad z = -h(x,y,t)\,, \tag{2.1}$$

which is combined with a sediment flux law

$$h_t = \nabla.\mathbf{Q}\,, \quad \text{at} \quad z = -h(x,y,t)\,, \tag{2.2}$$

where $\mathbf{Q}$ is the sediment flux. In the literature there are many flux laws proposed, depending on the assumed sediment type (see, for instance [17], [18], [19] and others). Here we follow the concepts used by [19] in particular, but with some simplifications. Thus

$$\mathbf{Q} = \frac{\mathbf{1}}{\mathbf{1-p_s}}(\mathbf{Q_b}+\mathbf{Q_s})\,.$$

note here we put $\mu = 1$ as we assume that the waves are always breaking where $p \approx 0.4$ is the bed porosity. The quantities $\mathbf{Q_{b,s}}$ are the bed-load and suspended sediment fluxes respectively, and are given by expressions of the form

$$\mathbf{Q_b} = \nu_\mathbf{b}(|\mathbf{u_w}|^\mathbf{2}\mathbf{u} - \lambda_\mathbf{b}|\mathbf{u_w}|^\mathbf{3}\nabla\mathbf{h})\,,$$

$$\mathbf{Q_s} = \nu_\mathbf{s}(\mathbf{H}|\mathbf{u_w}|^\mathbf{3}\mathbf{u} - \lambda_\mathbf{s}|\mathbf{u_w}|^\mathbf{5}\nabla\mathbf{h})\,.$$

Here $|u_w|$ is the wave velocity magnitude, $\nu_b$ and $\nu_s$ are coefficients of bed-load and suspended transport respectively. Usually these expressions are used outside the surf zone, but here we assume that they remain valid, at least in qualitative form, inside the surf zone. Then we assume that $|u_w| \approx |\mathbf{u}|$ and ignore the terms in $\nabla h$ as, in our theory, the beach slope is assumed small. Assuming also that in the surf zone $|\mathbf{u}| \approx \mathbf{2}\sqrt{\mathbf{gH}}$ (the simple wave expression) we finally get that

$$\mathbf{Q} = \mathbf{C}|\mathbf{u}|^\beta\mathbf{u}\,. \tag{2.3}$$

Here the coefficient $\beta$ varies between 2 and 5, depending on whether the bed-load or suspended sediment flux term dominates. The constant $C$ similarly then varies from $\nu_b/1-p_s$ and $\nu_s/4g(1-p_s)$. We choose $\beta = 3$ for analytical convenience, and basing an estimate for $C$ on $\nu_b(= 4\times10^{-5}s^2m^{-1})$, we set $C = \nu_b/u_b(1-p_s)$ where $u_b$ is a suitable velocity scale, say $u_b = \sqrt{gh_b}$.

The mass equation holds as before, but now allowing for the time dependence in $h$,

$$H_t + \nabla.(\int_{-h}^{\zeta}\mathbf{u})dz\ = 0\,, \quad \text{where} \quad H = h + \zeta\,. \tag{2.4}$$

Now, as before, let all quantities be written as a mean plus a fluctuation, so that for instance $h = \bar{h} + h'$. But because $\mathbf{Q}$ is nonlinear, we see that $h'$ is at least second order in wave-amplitude, and hence can be neglected. It may be necessary to examine this hypothesis later. Proceeding, we can average the new equations (2.2 , 2.4) to get

$$\bar{H}_t + \nabla.(\bar{H}\bar{\mathbf{u}}) = 0\,, \tag{2.5}$$

where $\bar{H} = \bar{h} + \bar{\zeta}$, together with

$$\bar{h}_t\ =\ \nabla.\bar{\mathbf{Q}}\,, \tag{2.6}$$

$$\text{where} \quad \bar{Q} = C < |\mathbf{u}|^3\mathbf{u} >\,, \tag{2.7}$$

and now needs to be evaluated as a function of both the mean and fluctuating components.

The momentum balance equation

$$\frac{\partial\bar{\zeta}}{\partial t} + \frac{\partial}{\partial x_i}(\bar{H}U_i) = 0 \tag{2.8}$$

$$\bar{H}\frac{\partial U_i}{\partial t} + \bar{H}U_j\frac{\partial U_i}{\partial x_j} = -\frac{\partial}{\partial x_j}S_{i\,j} - g(\bar{\zeta}+h)\frac{\partial\bar{\zeta}}{\partial x_j} \tag{2.9}$$

is still essentially but with the modified $H$ as above, that is, again henceforth omitting the "overbar" on all mean quantities henceforth for convenience,

$$H\frac{\partial U_i}{\partial t} + HU_j\frac{\partial U_i}{\partial x_j} = -\frac{\partial S_{ij}}{\partial x_j} - gH\frac{\partial \zeta}{\partial x_j}. \tag{2.10}$$

Altogether three equations for $\bar{\zeta}, \bar{h}, \mathbf{U}$, where the new feature is that $\bar{h}$ is a new unknown, and so there as new equation for it. Now we reexamine the wave-setup and longshore current problems, taking account of this new sediment term only inside the surf zone. We now seek solutions of (2.8 & 2.9 ) which are steady, and do not depend on the longshore coordinate. Thus all variables depend on $x$ only. For convenience here we shall omit the "overbar", that is "$\bar{U} = U$ and so on. Thus (2.8) immediately implies that U =0.

# 3 Shoaling Zone

Here we express, as before, the horizontal velocity $\mathbf{u} = (u, v)$ as a mean plus a fluctuation, thus

$$u = u' + \bar{u}\,,$$

for instance, where $u'$ is $O(a)$ and $\bar{u}$ is $O(a^2)$. Indeed here in the shoaling zone $u' \propto \sin\theta, \theta = kx + ly - \omega t$. But the averaging of

$$< |\sin\theta|^3 \sin\,\theta >= \int_0^{\pi} \sin^4\theta\, d\theta + \int_{\pi}^{2\pi} -\sin^4\theta\, d\theta = 0,$$

and so outside the surf zone $\bar{Q} = 0$. Indeed, this result will hold for all values of the index $\beta$. Thus, to the amplitude order considered, there is no mean sediment flux in $x > x_b$, and hence no change to our previous results.

# 4 Wave Set Up: Surf Zone

Inside the surf zone, we assume that $|\mathbf{u}| \approx \mathbf{2\sqrt{gH}}$ (the simple wave expression) in the expression (2.7) to get that

$$Q_x = \delta_0\, H^2, \tag{4.1}$$

$$\text{where} \quad \delta_0 = 16g^2C\,.$$

Also in the $y$ direction, since $v \approx u\tan\theta$,and since $\theta < 1$,

$$Q_y = \delta_0\, H^2\tan\theta. \tag{4.2}$$

We note that $\delta_0$ has dimensions of inverse time, and setting the velocity scale $u_b = \sqrt{gh_b} = 4.43ms^{-1}$ say , corresponding to $h_b = 2m$, we find that $\delta_0 = 2.32 \times 10^{-2}s^{-1}$.

We now examine the effect of this extra effect on wave set-up in the surf zone. Thus the momentum balance between the setup and flux term is again found from the mean momentum equation (2.10), keeping only the leading order terms,

$$\frac{\partial S_{11}}{\partial x} + g\,H\frac{\partial\zeta}{\partial x} = 0. \tag{4.3}$$

This is solved exactly as in Chapter 3, since as before $S_{11} = 1/2\,\Gamma g\,H^2$, but note that $\Gamma = 3\gamma^2/8$ so that

$$\Gamma g H H_x + gH(H_x - \bar{h}_x) = 0\,.$$
$$\text{and so} \quad H_x[1+\Gamma] = \bar{h}_x. \tag{4.4}$$

Integration gives that

$$H[1+\Gamma] = \bar{h} + C(t)\,. \tag{4.5}$$

In general $C = C(t)$ but here we impose the boundary condition that $\bar{h}_b = h_b$ is a constant. Then $C > 0$ is a constant, found from the matching at the edge of the surf zone, $x = x_b$.

$$[1+\Gamma]H_b = \bar{h}_b + C\,.$$
$$\text{or} \quad C = \Gamma h_b + [1+\Gamma]\zeta_b\,.$$

Next, use the sediment equation (2.6) to get

$$\bar{h}_t = 2\delta_0 H H_x\,. \tag{4.6}$$

Hence, eliminating $\bar{h}$ we get

$$\bar{h}_t = M(\bar{h} + C)\bar{h}_x\,, \quad M = \frac{2\delta_0}{(1+\Gamma)^2}\,. \tag{4.7}$$

The initial condition is that $\bar{h} = h(x), x < x_b$ at $t = 0$, and, as above, there is a boundary condition that $\bar{h} = h_b$ at $x = x_b, t > 0$. Hence the solution, found using characteristics is

$$\frac{dx}{dt} = -M(C + \bar{h}),\ \frac{d\bar{h}}{dt} = 0\,, \quad \text{so that} \tag{4.8}$$

$$\bar{h} = h(x + MCt + M\bar{h}t)\,, \quad \text{for} \quad x + MCt + Mh_bt < x_b\,, \tag{4.9}$$
$$\bar{h} = h_b\,, \quad \text{for} \quad x + MCt + Mh_bt > x_b\,, \tag{4.10}$$

However, the characteristics (4.8) from $x < x_b$ intersect and a shock forms, eventually making the solution invalid.

For example, for a linear beach, $h = \alpha x$ we get

$$\bar{h} = \frac{\alpha(x + MCt)}{1 - \alpha Mt}\,, \quad \text{for} \quad x + M(C + \alpha x_b)t < x_b\,.$$

The solution blows up as $\alpha Mt \to 1$. Before that the depth increases and the shore line where $\bar{h} = 0$ recedes. The total depth is now given by equation (4.5), but we recall that it is only valid for small time, $Mt << 1$. For the linear depth profile, this is

$$H(1+\Gamma) = \frac{\alpha x + C}{1 - \alpha Mt}\,. \tag{4.11}$$

Note that the shoreline $x = x_s$ is given by $\alpha x_s = -C$, which is unaffected by the sediment transport. But the total depth increases uniformly at all locations as $\alpha Mt$ increases. Here, setting $\gamma = 0.88$, we find that $M = 2.79 \times 10^{-2} s^{-1}$; thus for a beach slope of $\alpha = 0.1$, we see that the time scale for the sediment transport to take effect is $1/\alpha M = 360\,s$. Although we have made several simplifying approximations, this estimate seems quite reasonable, being about $30 - 40$ wave periods.

Next, we now see from equation (2.5) that $\bar{U} \neq 0$ and so one must use the new expression for $H$, that is $H_t$ to estimate $\bar{U}$. In turn this is then a correction to equation (2.10). So the set-up can be re-calculated, as described in the next section.

# 5 Wave Set Up: Surf Zone, Second Iteration

With $\bar{h}, H$ now determined as above, return to the mass conservation equation (2.5) to find the mean velocity $U$ (overbar omitted here)

$$H_t = -(HU)_x\,, \quad \text{or} \quad \frac{2\delta_0 H H_x}{(1+\Gamma)} = -(HU)_x\,. \tag{5.1}$$

Assuming that $HU = 0$ at the shoreline where $H = 0$ we get

$$U = -\frac{\delta_0 H}{(1+\Gamma)}\,. \tag{5.2}$$

This expression does imply a non-zero onshore flow at $x = x_b$ and so strictly an onshore flow is now needed in $x > x_b$ as well. Since at $x = x_b$, this flow is independent of $t$ it can be found also from (2.5) by looking for a time-independent solution, which is

$$HU = \text{constant} = H_b U_b = -\frac{\delta_0 H_b^2}{(1+\Gamma)}\,. \tag{5.3}$$

We now return to the momentum equation, where we see that the term $H\partial U/\partial t$ is not zero, and can be found from (5.2). Thus (4.3) is replaced by

$$\frac{\partial S_{11}}{\partial x} + g\,H\frac{\partial \zeta}{\partial x} + H\frac{\partial U}{\partial t} = 0\,.$$

Using (5.2) we get

$$\frac{\partial S_{11}}{\partial x} + g\,H\frac{\partial \zeta}{\partial x} = \frac{\delta_0 H H_t}{(1+\Gamma)} = \delta_0 M H^2 H_x\,. \tag{5.4}$$

This can now be integrated to give

$$H(1+\Gamma) \;=\; \bar{h} + C + \frac{\delta_0 M}{2g}H^2\,. \tag{5.5}$$

As before $C$ is a constant found by putting $H = H_b$ at $x = x_b$. The last term can be approximated using the first iteration to give

$$H(1+\Gamma) \;=\; \bar{h} + C + \frac{M^2}{2g}(\bar{h}+C)^2\,. \tag{5.6}$$

This can now be substituted into the sediment equation (4.6) to get a correction to (4.7),

$$\bar{h}_t = M(\bar{h} + C + \frac{3M^2}{2g}(\bar{h}+C)^2)\bar{h}_x\,. \tag{5.7}$$

With the same initial conditions and boundary conditions as before, this can be integrated as before by characteristics.

$$\frac{dx}{dt} = -M(C+\bar{h}) - \frac{3M^3}{2g}(\bar{h}+C)^2,\ \frac{d\bar{h}}{dt} = 0\,, \quad \text{so that} \tag{5.8}$$

$$\bar{h} \;=\; h(x + (\bar{h}+C)Mt + (\bar{h}+C)^2\frac{3M^3 t}{2g})\,, \quad \text{for} \quad x + MCt + Mh_b t < x_b\,, \tag{5.9}$$

$$\bar{h} \;=\; h_b\,, \quad \text{for} \quad x + MCt + Mh_b t > x_b\,, \tag{5.10}$$

Now, for a linear depth, $h(x) = \alpha x$ we get a quadratic equation for $\bar{h}$. But as the quadratic term is a small correction, we can simplify to get that

$$\bar{h} \approx \frac{\alpha(x + CMt)}{1 - \alpha Mt} + \frac{3M^3 \alpha t(\alpha x + C)^2}{2g(1 - \alpha Mt)^2}\,, \tag{5.11}$$

$$\text{and} \quad H(1 + \Gamma) \approx \frac{\alpha x + C}{1 - \alpha Mt} + \frac{M^2(\alpha x + C)^2(1 + 3\alpha Mt)}{2g(1 - \alpha Mt)^2}\,. \tag{5.12}$$

Thus the correction term acts to increase the depth $\bar{h}$ and to move the shoreline onshore.

# 6 Wave Set Up: Surf Zone, Simple Wave Solution

The basic equations for set-up, with only $x, t$ dependence are given by equations (2.8) and (2.9) with use of the sediment law equation (2.6), these are

$$H_t + (HU)_x = 0\,, \tag{6.1}$$

$$\bar{h}_t - 2\delta_0 H H_x = 0\,, \tag{6.2}$$

$$U_t + UU_x + gH_x - g\bar{h}_x + \Gamma g H_x = 0\,. \tag{6.3}$$

This is a $3 \times 3$ nonlinear hyperbolic system. This can be written in the form

$$\mathbf{v}_t + \mathbf{A}(\mathbf{v})\mathbf{v}_x = 0\,, \quad \text{where} \quad \mathbf{v}_t = [H, U, \bar{h}]\,, \tag{6.4}$$

and the $3 \times 3$ matrix $\mathbf{A}$ is given by

$$\mathbf{A} = [U, H, 0], [g(1 + \Gamma), U, -g], [-2\delta_0 H, 0, 0]. \tag{6.5}$$

The eigenvalues $\lambda$ of $\mathbf{A}$ are given by

$$\det[\mathbf{A}(\mathbf{v}) - \lambda \mathbf{I}] = 0\,, \tag{6.6}$$

which leads to the cubic equation

$$\lambda[g(1 + \Gamma)H - (U - \lambda)^2] + 2g\delta_0 H^2 = 0\,. \tag{6.7}$$

The system is hyperbolic if this has three real roots. For small $\delta_0$ the roots are

$$\lambda_1 \approx -\frac{2g\delta_0 H}{[g(1 + \Gamma)H - U^2]}\,, \quad \lambda_{2.3} \approx U \pm [g(1 + \Gamma)H]^{1/2}\,. \tag{6.8}$$

All are real-valued, and so in this limit the system is hyperbolic. The first root is the one of main interest here, as it is due directly from the sediment transport term. More generally, we can show that all the roots are real-valued, and so the system is hyperbolic, provided that

$$27g\delta_0 H^2 < U^3 + [U^2 + 3g(1 + \Gamma)H]^{3/2} - 9Ug(1 + \Gamma)H. \tag{6.9}$$

We assume henceforth that this condition is always satisfied. Note that the left-hand side is greater than zero for all values of $U$ and all $H > 0$, except when $U = [g(1 + \Gamma)H]^{1/2}$ when it is zero.

The previous small-time perturbation solution suggests that we seek a simple-wave solution in the form

$$\mathbf{v} = \mathbf{v}(\alpha)\,, \tag{6.10}$$

where $\alpha = \alpha(x, t)$ is an arbitrary new variable, and could be taken as any one of the set $H, U, \bar{h}$. Substituting into (6.4) shows that

$$\alpha_t = c(\alpha)\alpha_x\,, \quad \text{where} \quad -c = \lambda \tag{6.11}$$

is one of the eigenvalues of $\mathbf{A}$, and $\mathbf{v}_\alpha$ is then a corresponding eigenvector, so that for instance

$$(U-\lambda)H_\alpha + HU_\alpha = 0\,, \tag{6.12}$$
$$2\delta_0 HH_\alpha + \lambda = 0\,. \tag{6.13}$$

We choose $\lambda = \lambda_1$, the root corresponding to the sediment transport term, and given approximately by (6.8) when $\delta_0$ is small. In this limit, we choose $\alpha = \bar{h}$ and readily find that

$$c = \frac{2\delta_0(\bar{h}+C)}{(1+\Gamma)^2} + \frac{3\delta_0^3(\bar{h}+C)^2}{g(1+\Gamma)^6} + \cdots\,, \tag{6.14}$$
$$U = -\frac{\delta_0(\bar{h}+C)}{(1+\Gamma)^2}\,, \tag{6.15}$$
$$H = \frac{(\bar{h}+C)}{(1+\Gamma)} + \frac{\delta_0^2(\bar{h}+C)^2}{2g(1+\Gamma)^5} + \cdots\,. \tag{6.16}$$

The leading order expressions agree with those found before.

Returning to the simple wave equation (4.4), the solution can again be found by the method of characteristics, that is

$$\frac{dx}{dt} = -c(\bar{h}),\ \frac{d\bar{h}}{dt} = 0\,, \quad \text{for} \quad x < x_b\,, \tag{6.17}$$
$$\bar{h} = h_b \quad \text{at} \quad x = x_b\,, \tag{6.18}$$
$$\text{so that} \quad x + c(\bar{h})t = x_0\,, \bar{h} = h(x_0) \quad \text{for} \quad x_0 < x_b\,, \tag{6.19}$$

where $x_0$ is the initial value of $x$ along each characteristic. This can then be rewritten in the form

$$\bar{h} = h(x + c(\bar{h})t)\,, \quad \text{for} \quad x + c(h_b)t < x_b\,, \tag{6.20}$$
$$\bar{h} = h_b \quad \text{for} \quad x + c(h_b)t < x_b\,, \tag{6.21}$$

It follows that if $c(\bar{h})$ is an increasing function of $\bar{h}$, that is $c_{\bar{h}} > 0$, then the characteristics will intersect, and a shock, indicating a breakdown of this simple wave solution. In turn this implies a breakdown of the present sediment transport model. The approximate expression (6.14) indicates that $c_{\bar{h}} > 0$ at least for sufficiently small $\delta_0$. We also note that $c(\bar{h})$ is independent of the initial profile $h(x)$ and the breakdown will therefore occur for all beach profiles, provided only that $h_x > 0$.

# 7 Wave Set Up: Surf Zone, Steady State

Indeed, it is already clear from (6.2) that the present sediment transfer model cannot allow any steady state to form, as $\bar{h}_t = 0$ would then imply that $H = 0$, which is unacceptable. Hence, if a steady state is to be reached, we must replace the sediment law (2.3) by an expression which takes account of the actual beach slope. Thus, from the discussion in section 5.1, we now replace (2.3) with

$$\mathbf{Q} = \mathbf{C}|\mathbf{u}|^{\beta}\mathbf{u} - \mathbf{D}|\mathbf{u}|^{\beta+2}\nabla\mathbf{h}\,. \tag{7.1}$$

Here $D$, like $C$, is an empirical constant. Choosing $\beta = 3$ as before, and averaging we now replace (4.1) with (again omitting the "overbar")

$$Q_x = \delta_0\, H^2 - \delta_1 H^{5/2}\bar{h}_x\,. \tag{7.2}$$

Here $\delta_1$ is an empirical constant, whose value we estimate to be $\delta_1 = (2\,g)^{1/2}\,0.7\delta_0 m^{-1/2}$. Substituting this into (2.6) we get instead of (6.2)

$$\bar{h}_t = 2\delta_0 HH_x - \delta_1(H^{5/2}\bar{h}_x)_x\,. \tag{7.3}$$

This is a nonlinear diffusion equation, and so there is a possibility that a steady-state can be achieved. Indeed if we assume that there is a steady-state solution then (7.3) implies that $Q_x = 0$, so that

$$\delta_1 H^{1/2} \bar{h}_x = \delta_0 \,, \tag{7.4}$$

where a constant of integration has been set to zero since $H = 0$ at the shore line. Further, in a steady-state, (6.1) implies that $U = 0$ and (6.3) can be integrated to yield

$$[1 + \Gamma]H = \bar{h} + C \,, \tag{7.5}$$

compare (4.5), where $C$ is a constant, that is

$$[1 + \Gamma]H_b \,=\, \bar{h}_b + C \,.$$

Substituting into (7.4) and integrating we find that

$$\frac{2[1 + \Gamma]\delta_1 H^{3/2}}{3\delta_0} = x - x_s \,, \quad \text{where} \quad \frac{2[1 + \Gamma]\delta_1}{3\delta_0} H_b^{3/2} = x_b - x_s \,. \tag{7.6}$$

Here $x = x_s$ is the shoreline. Thus in the equilbrium state, this model predicts that the total depth follows the power law $(x - x_s)^{2/3}$. Remarkably this is precisely the famous Dean's law! ([20],[21]).

## 7.1 Long-shore transport

Assuming as above that we can ignore the time-dependence at the leading order, we recall that the momentum balance

$$\nu_0 \frac{dV}{dx} = B_0$$

where $B_0 = \frac{1}{8} g\gamma^2 \cos\theta_b$ yields, that is,

$$\nu_0 V = B_0(x - x_s), \tag{7.7}$$

where $B_0$ is some known constant.The only effect comes from the shoreline $x_s$, which is now time-dependent.

## 7.2 Sediment controlled rip currents in surf zone

For the rip current model see [22], the shoaling zone remains unchanged. But in the surf zone, we need to take account of the sediment transport. Assuming that the effect of the sediment transport is small, that is as above, assume that $Mt << 1$, we can estimate the effect as follows. First we recall that the steady-state equations are

$$H[U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y}] = -g\, H\frac{\partial \zeta}{\partial x} - [\tau_x], \tag{7.8a}$$

$$H[U\frac{\partial v}{\partial x} + V\frac{\partial V}{\partial y}] = -g\, H\frac{\partial \zeta}{\partial y} - [\tau_y], \tag{7.8b}$$

while the vorticity equation is

$$\psi_x(\frac{\Omega}{H})_y - \psi_y(\frac{\Omega}{H})_x = [\frac{\tau_x}{H}]_y - [\frac{\tau_y}{H}]_x, \tag{7.9}$$

where $\Omega$ is define as

$$\Omega = V_x - U_y = (\frac{\psi_x}{H})_x + (\frac{\psi_y}{H})_y \,. \tag{7.10}$$

Previously, we approximated $H$ with $h(x)$. Here we now retain the full $H$ given by (4.5) (for the case of a linear depth profile), noting again that although this has a small time-dependence, we shall nevertheless continue to use the steady-state equations above.

For this case of a linear depth, $H$ is given by (4.11), and we see that in effect, the only change due to the sediment transport is that we have in effect replaced $H = \alpha x$ by

$$H = \frac{\alpha(x - x_s)}{(1 + \Gamma)(1 - \alpha M t)} .$$

Apart from the change of origin, noting that $x_s$ does not depend on $t$, the most significant consequnce is in effect the slope has changed form $\alpha$ to $1/(1 + \Gamma)(1 - \alpha M t)$. Here the first factor in the denominator acts to reduce the slope independently of the sediment transport, but the second factor which is due to the sediment transport, then acts to increase the slope. As discussed in [22] an increase (decrease) in slope decreases (increases) the ratio parameter $R$, which in turn decreases the rip current circulation *vis-a-vis* the longshore current field.

# 8 Conclusion

We examined a simple model of sediment transport, induced by the breaking waves in the surf zone. Essentially the bottom is allowed to move in response to the divergence of a sediment flux, in turn determined by the breaking waves. The effect of this extra term on the previous solutions for set-up, longshore currents and rip currents is then determined. It is found that the solutions for the mean flow are now unsteady on a slow timescale determined by a certain sediment transport parameter. There is a change in beach slope in the rip currents controlled by the sediment transport. The system of equations now forms a three-by-three nonlinear hyperbolic system of equations. These we solve approximately, using a simple wave solution based on the simple wave speed corresponding to the small sediment transport parameter. However, this solution will always breakdown after a long time, so we show that by adding another term proportional to the beach slope into the expression for the sediment flux, we can obtain a steady-state solution.

# Competing Interests

Author has declared that no competing interests exist.

# References


[1] Longuet-Higgins MS, Stewart, R.W. The changes in the form of short gravity waves on long waves and tidal currents. Journal of Fluid Mechanics. 1960;8:565-583.

[2] Longuet-Higgins MS, Stewart RW. The changes in amplitude of short gravity waves on on steady non-uniform currents. Journal of Fluid Mechanics. 1961;10:529-549.

[3] Longuet-Higgins MS, Stewart RW. Radiation stress and mass transport in gravity waves, with application ro surf beats. Journal of Fluid Mechanics. 1962;13:481-504.

[4] Longuet-Higgins MS, Stewart RW. Radiation stress in water waves: A physical discussion with applications. Deep-sea Research. 1964;11:529-562.

[5] Longuet-Higgins MS. Longshore currents generated by obliquely incident sea waves, 1. J. Geophysical Research.1970;75:6778-6789.

[6] Longuet-Higgins MS. Longshore currents generated by obliquely incident sea waves, 2. J. Geophysical Research. 1970;75:6790-6801.

[7] Battjies JA. Surf zone dynamics. Annual Rev. Fluid Mechanics. 1988;20:257-293.

[8] List JH. A model for the generation of two dimensional surf beat. Journal of Geophysical Research. 1992;97:5623-5635.

[9] Herbers THC, Elgar S, Guza RT. Infragravity- frequency $(0.005 - 0.05Hz)$ motions on the shelf, I. Forced waves. Journal of Physical Oceanography. 1994;24:917-927.

[10] Schaffer HA. Infragravity waves induced by short wave groups. Journal of Fluid Mechanics. 1993;247:551-588.

[11] Symonds G, Huntley DA, Bowen AJ. Two-dimensional surf beat: Long wave generation by a time-varying breakerpoint; 1982.

[12] Munk WH. Surf beats. Translation of American Geophysical Union. 1949;30:849-854.

[13] Wright LD, Guza RT, Short AD. Dynamics of a high energy dissipative surf zone. Marine Geology. 1982;45:41-62.

[14] Tucker MJ. Surf beats: Sea waves of 1 to 5 min. period. Proc. R. Soc. London. 1950;202:565-573.

[15] Huntley DA, Guza RT, Thornton EB. Field observations of surf beat, 1, Progressive edge waves. Journal of Geophysical Research. 1981;86:6451-6466.

[16] Mei CC. Applied dynamics of ocean surface waves. John Wiley and Sons, Inc; 1983.

[17] Calvete D, Falques A, De Swart HE, Walgreen M. Modelling the formation of shoreface-connected sand ridges on storm-dominated inner shelves. Journal of Fluid Mechanics. 2001;441:169-193.

[18] Restrepo JM. Wave-current interactions in shallow waters and shore-connected ridges. Continental Shelf Research. 2001;21:1331-1360.

[19] Lane EM, Restrepo JM. Shoreface-connected ridges under the action of waves and currents. Journal of Fluid Mechanics. 2007;582:23-52.

[20] Dean, Dalrymple. Coastal processes with engineering applications. Cambridge University Press; 2002.

[21] Dean RG. Equilibrium beach profiles: Characteristics and applications.' Journal of Coastal Research. 1991;7(l):53-84.

[22] Osaisai EF. Modeling the rip current flow on a quadratic beach profile. International Journal of Computational Engineering Research. 2013;3(10)

---



***Peer-review history:***
*The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)*
*http://sciencedomain.org/review-history/19865*