British Journal of Mathematics & Computer Science
15(1): 1-8, 2016, Article no.BJMCS.24577
ISSN: 2231-0851

SCIENCEDOMAIN *international*
www.sciencedomain.org

On Finding Geodesic Equation of Two Parameters Inverse Gaussian Distribution

William W. S. Chen[1*]

[1]*Department of Statistics, The George Washington University, Washington D.C. 20013, USA.*

Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/BJMCS/2016/24577
Editor(s):
(1) Morteza Seddighin, Indiana University East Richmond, USA.
Reviewers:
(1) Jose Rosales-Ortega, University of Costa Rica, Costa Rica.
(2) Anonymous, UAM Cuajimalpa, Mexico.
Complete Peer review History: http://sciencedomain.org/review-history/13529

Method Article

Received: 26th January 2016
Accepted: 22nd February 2016
Published: 3rd March 2016

Abstract

The class of Inverse Gaussian distributions is quite commonly used as a life-time model in reliability studies. The books by Chhikara and Folks [1] and Seshadri [2] present extensive discussions on classical inference for the parameters of Inverse Gaussian distribution.

However, in this paper, we switch our attention to find its geodesic equation. We applied two different algorithms to solve some partial differential equations, where these equations originated from the Inverse Gaussian distribution. As expected, the two algorithms yield the same result.

Keywords: Darboux theory; differential geometry; geodesic equation; inverse Gaussian distribution; triply partial differential equation.

Mathematical subject classification: 62E99.

1 Introduction

The inverse Gaussian distribution is particularly important to probabilists and physicists due to its relation to Brownian motion. Balakrishnan N. and Chen W.W.S. [3] have completed the tremendous task of computing the means, variances and covariances of order statistics for all sample sizes up to twenty five and for many

**Corresponding author: E-mail: williamwschen@gmail.com;*

choices of the shape parameter. The major reason for taking up this study is to make use of the tabulated values of means, variances and covariances of order statistics in order to derive the best linear unbiased estimators of the location and scale parameters based on complete as well as Type-II censored samples. Aside from the theoretical issues, the distribution has much to offer to the practicing statistician in many important areas of application. For example, the pioneering work of Tweedie [4] used the inverse Gaussian in a clinical trial study of the effect of a drug on the first passage time taken for a jejunal biopsy capsule, on leaving the stomach, to travel from the pylorus through the duodenum and into the jejunum. Working in collaboration with statisticians in the Clinical Cancer Research Institute in Liverpool, Tweedie studied the distribution of survival times in a series of patients who had been treated for cancer, and fitted the inverse Gaussian with considerable success. He also found that the antilognormal and Weibull were poor fits. In this paper, instead of studying all previously stated topics of interest, we switch to a new direction, by finding the geodesic equation of an inverse Gaussian distribution. We used two different algorithms to approach this purpose, and found that both algorithms reach the same result. In section 2, we list the fundamental tensor we need for later use. In section 3, we present two algorithms to show the process of deriving the required geodesic equation. In section 4, we repeat some more detail procedures on how the fundamental tensor has been derived. Six Christoffel symbols have been collected in the appendix. There are many related books published in this area of study, including Kass R.E. and Vos P.W. [5] and Amari S-I [6].

2 List the Fundamental Tensor

The standard or canonical form of the two-parameter Inverse Gaussian distribution has the probability density function given by,

$$f(y,u,v) = \left(\frac{v}{2\pi y^3}\right)^{\frac{1}{2}} \exp\left(-\frac{v}{2u^2 y}(y-u)^2\right) \qquad y > 0$$

$$\ln f(y,u,v) = \frac{1}{2}(\ln v - \ln(2\pi y^3)) - \frac{v}{2u^2 y}(y-u)^2$$

It is known that, in this form of the distribution, u is the mean and $\frac{u^3}{v}$ is the variance, Furthermore, $\frac{u}{v}$ is the square of the coefficient of variation. From the equation above, we derive the metric tensor components for this distribution as follows,

$$E = -E\left(\frac{\partial^2 \ln f(x)}{\partial u^2}\right) = \frac{v}{u^3}, \qquad F = -E\left(\frac{\partial^2 \ln f(x)}{\partial v \partial u}\right) = 0,$$

$$G = -E\left(\frac{\partial^2 \ln f(x)}{\partial v^2}\right) = \frac{1}{2v^2}$$

Using the results above, we can easily derive the required basic tensor metric and Christoffel symbols as follows:

$$E_u = \frac{-3v}{u^4}, \quad E_v = \frac{1}{u^3}, \; F_u = 0, \; F_v = 0, \quad G_u = 0,$$

$$G_v = \frac{-1}{v^3}, \quad EG = \frac{1}{2u^3 v}$$

$$\Gamma_{11}^1 = \frac{E_u}{2E} = \frac{-3}{2u}, \quad \Gamma_{12}^2 = \frac{G_u}{2G} = 0 \quad \Gamma_{11}^2 = \frac{-E_v}{2G} = \frac{-v^2}{u^3},$$

$$\Gamma_{22}^1 = \frac{-G_u}{2E} = 0 \quad \Gamma_{12}^1 = \frac{E_v}{2E} = \frac{1}{2v}, \quad \Gamma_{22}^2 = \frac{G_v}{2G} = \frac{-1}{v}$$

3 The Geodesic Equation

One method to find the geodesic equation of the Inverse Gaussian distribution is by solving a triply of partial differential equations given in the appendix (see Struik, D.J. or Grey, A [7,8]). We seek its solution as below. To avoid confusion, we will only index those formulas that we will use later, and we will ignore the others:

$$\frac{d^2u}{ds^2}-\frac{3}{2u}\left(\frac{du}{ds}\right)^2+\frac{1}{v}\frac{dudv}{dsds}=0, \tag{1}$$

$$\frac{d^2v}{ds^2}-\frac{v^2}{u^3}\left(\frac{du}{ds}\right)^2-\frac{1}{v}\left(\frac{dv}{ds}\right)^2=0, \tag{2}$$

And the first fundamental form of distance function is given

by
$$\begin{aligned}ds^2 &= Edu^2+2Fdudv+Gdv^2\\ &=\frac{v}{u^3}du^2+\frac{1}{2v^2}dv^2\end{aligned} \tag{3}$$

It needs only two out of the three equations above to find the geodesic equation. We will choose equations (1) and (3). To simplify the notation, we let

$$\begin{aligned}&\frac{du}{ds}=p, \quad \frac{d^2u}{ds^2}=\frac{dp}{ds} \quad \text{then}\\ &\frac{dp}{ds}-\frac{3}{2u}p^2+\frac{p}{v}\frac{dv}{ds}=0.\end{aligned} \tag{4}$$

Dividing the equation (4) by p, and integrating on both sides with respect to p, we get

$$\frac{\frac{dp}{ds}}{p}-\frac{3}{2u}p+\frac{1}{v}\frac{dv}{ds}=0, \qquad \ln p-\frac{3}{2}\ln u+\ln v=C$$

$$or \quad \ln p = C+\frac{3}{2}\ln u-\ln v=\ln A+\ln\frac{u^{3/2}}{v}=\ln\frac{Au^{3/2}}{v}$$

Where we choose C=ln A

$$p=\frac{Au^{3/2}}{v}=\frac{du}{ds} \qquad \text{or} \qquad Ads=\frac{vdu}{u^{3/2}}$$

Square both side, we get

$$A^2ds^2=\frac{v^2du^2}{u^3} \tag{5}$$

From equation (3), we see that the first fundamental form of Inverse Gaussian Model is given by:

$$ds^2=\frac{v}{u^3}du^2+\frac{1}{2v^2}dv^2$$

Hence, $A^2 ds^2 = \frac{A^2 v}{u^3} du^2 + \frac{A^2}{2v^2} dv^2$ (6)

Put (5) and (6) together we get

$$\frac{\text{v}^2}{u^3} du^2 = \frac{A^2 v}{u^3} du^2 + \frac{A^2}{2v^2} dv^2$$

$$\text{or } \frac{(\text{v}^2 - A^2 v)}{u^3} du^2 = \frac{A^2}{2v^2} dv^2 \quad (7)$$

By taking the square root of equation (7)

$$\pm\sqrt{\frac{(\text{v} - A^2)v}{u^3}} du = \frac{\pm A dv}{\sqrt{2}v}$$

$$\text{or } \frac{\pm du}{u^{3/2}} = \frac{\pm A dv}{\sqrt{2}v\sqrt{(\text{v} - A^2)v}}$$

Integrating both sides, we get

$$\pm\int u^{-3/2} du \pm \frac{1}{\sqrt{2}} \int \frac{A dv}{v\sqrt{(\text{v} - A^2)v}} = B$$

$$or \ \pm\frac{2}{\sqrt{u}} \pm \frac{1}{\sqrt{2}} \int \frac{A dv}{v^{3/2}\sqrt{\text{v} - A^2}} = B$$

Where A, B are arbitrary constants.

Alternatively, we can find the geodesic equation of the Inverse Gaussian distribution by solving one partial differential equation. This idea originated from French mathematician Darboux's theory. A detailed proof has been given in Chen [9,10]. From section 2, we know that the coefficient of the first fundamental form of $\nabla Z = 1$ is given by,

$$E = -E\left(\frac{\partial^2 \ln f(x)}{\partial u^2}\right) = \frac{v}{u^3}, \qquad F = -E\left(\frac{\partial^2 \ln f(x)}{\partial v \partial u}\right) = 0,$$

$$G = -E\left(\frac{\partial^2 \ln f(x)}{\partial v^2}\right) = \frac{1}{2v^2} \quad \Delta z = 1 \ ; \quad \frac{\text{v}}{\text{u}^3} z_v^2 + \frac{1}{2v^2} z_u^2 = \frac{1}{2u^3 v}$$

To solve the partial differential equation above, we may use the separable variable method as follows:

By multiple above equation $2\text{u}^3\text{v}$, we derived

$$2v^2 z_v^2 + \frac{u^3}{v} z_u^2 = 1$$

$$or \ u^3 z_u^2 = v(1 - 2v^2 z_v^2) = A^2 \ ,$$

Break this into two parts, we get

Part 1, $u^3 z_u^2 = A^2$ implies $z_u^2 = \frac{A^2}{u^3}$

$$or \;\; z_u = \pm\frac{A}{u^{3/2}} = Au^{-3/2} \;\; or \;\; Z = \pm 2Au^{-1/2} = \frac{\pm 2A}{\sqrt{u}} \qquad (8)$$

and part 2, $v(1-2v^2 z_v^2) = A^2$

$or \;\; v - A^2 = 2v^3 z_v^2$ so $z_v^2 = \frac{v - A^2}{2v^3}$

$$z = \pm\frac{1}{\sqrt{2}}\int\frac{\sqrt{v-A^2}}{v^{3/2}}dv \qquad (9)$$

The general solution of the geodesic equation combined (8) and (9), hence we get

$$z = \frac{\pm 2A}{\sqrt{u}} \pm \frac{1}{\sqrt{2}}\int\frac{\sqrt{v-A^2}}{v^{3/2}}dv$$

Applying the Darboux Theory, we finally find that the geodesic equation of Inverse Gaussian Distribution is given by

$$\frac{\partial z}{\partial A} = B \;\text{ or }\; \pm\frac{2}{\sqrt{u}} \pm \frac{1}{\sqrt{2}}\int\frac{Adv}{v^{3/2}\sqrt{v-A^2}} = B$$

Where A and B are arbitrary constants. This result is the same as the previous one.

4 Deriving the Basic Tensor

The probability density function of two parameters of the Inverse Gaussian distribution is given by

$$f(y,u,v) = \left(\frac{v}{2\pi y^3}\right)^{\frac{1}{2}}\exp\left(-\frac{v}{2u^2 y}(y-u)^2\right) \qquad y > 0$$

$$\ln f(y,u,v) = \frac{1}{2}(\ln v - \ln(2\pi y^3)) - \frac{v}{2u^2 y}(y-u)^2$$

From the equation above, we derive the metric tensor components for the distribution as follows,

$$\frac{\partial \ln f}{\partial u} = -\frac{v}{2y}\frac{\partial}{\partial u}\left(\frac{y-u}{u}\right)^2 = \frac{v(y-u)}{u^3}$$

$$\frac{\partial^2 \ln f}{\partial u^2} = \frac{v(-3y+2u)}{u^4},$$

$$\frac{\partial^2 \ln f}{\partial v \partial u} = \frac{y-u}{u^3},$$

$$\frac{\partial \ln f}{\partial v} = \frac{1}{2v} - \frac{(y-u)^2}{2u^2 y}, \quad \frac{\partial^2 \ln f}{\partial v^2} = \frac{-1}{2v^2}$$

For the next step, we need to find the expectation of these second order partial derivatives. The answer is almost straightforward, and we just list them below:

$$E = -E\left(\frac{\partial^2 \ln f}{\partial u^2}\right) = \frac{v}{u^3},\ \mathrm{F} = -E\left(\frac{\partial^2 \mathrm{lnf}}{\partial v \partial \mathrm{u}}\right) = 0,\ \mathrm{G} = -\mathrm{E}\left(\frac{\partial^2 \ln f}{\partial v^2}\right) = \frac{1}{2v^2}$$

5 Conclusion and Remarks

Geodesic equation interested both mathematician and statistician. The history of geodesic equation begins with John Bernoulli's solution of the problem of the shortest distance between two points on a convex surface in year 1697~ 1698. His answer was that the osculating plane must always be perpendicular to the tangent plane. The name "geodesic equation" in its present meaning is, according to Stackel, due to J. Liouville, Journal de mathem 9. 1844, p401. To better understand the application in statistics, we will use the following example. Let us assume the most common and elementary situation as the first course in elementary statistics. We wish to test the hypothesis $H_0\ v = v_0$ *versus* $H_1\ v \neq v_0$ with unknown parameter u. If $X_1......X_m$ is a random sample of size m from an IG(u,v) distribution, the maximum likelihood estimators are $\hat{u} = \bar{X},\ \hat{\mathrm{v}} = \frac{m}{\sum_1^m \left(\frac{1}{X_i} - \frac{1}{\bar{X}}\right)}$ (see Villarroya and Oller [11]). Then the critical region of the Rao distance test is derived to be

$$C = \{(X_1......X_m)/(Q < Q_1) \cup (Q > Q_2)\}$$

$$where\ \mathrm{Q} = \frac{\mathrm{mV_0}}{\bar{\mathrm{v}}},\ \mathrm{Q_1} = me^{-\sqrt{2u_\alpha}},\ \mathrm{Q_2} = me^{\sqrt{2u_\alpha}},\ \mathrm{Q_1} < \mathrm{Q_2})$$

$$and\quad \mathrm{F(Q_1) - F(Q_2) = 1 - \alpha},\qquad \mathrm{Q_1 Q_2 = m^2}$$

$$Under\ \mathrm{H_0}\ \ \mathrm{Q} \sim \chi_{(\mathrm{m-1})}$$

As we can see that Rao distance test is similar to geodesic equation.

Competing Interests

Author has declared that no competing interests exist.

References

[1] Chhikara RS, Folks JL. The Inverse Gaussian distribution: Theory, methodology and applications. Marcel Dekker, New York; 1989.

[2] Seshadri V. The Inverse Gaussian distribution: A case study in exponential families. Clarendon Press, Oxford, England; 1993.

[3] Balakrishnan N, Chen WWS. CRC handbook of order statistics from inverse Gaussian distributions with applications. CRC Press, Baca Raton; 1997.

[4] Tweedie MCK. Inverse statistical variates. Nature. 1945;155:453.

[5] Kass RE, Vos PW. Geometrical foundations of asymptotic inference. John Wiley & Sons, Inc; 1997.

[6] Amari SI. Differential-geometrical methods in statistics. Springer, New-York; 1990.

[7] Struik DJ. Lectures on classical differential geometry. Second Edition. Dover Publications, Inc; 1961.

[8] Grey A. Modern differential geometry of curves and surfaces. CRC Press, Inc. Boca Raton; 1993.

[9] Chen WWS. A note on finding geodesic equation of two parameters gamma distribution. Applied Mathematics. 2013;5:3511-3517.
Available: www.scirp.org/journal/am

[10] Chen WWS. A note on finding geodesic equation of two parameter Weibull distribution. Theoretical Mathematics & Applications. 2014;4(3):43-52.

[11] Villarroya A, Oller JM. Statistical tests for the inverse Gaussian distribution based on Rao distance. Barcelona: University of Barcelona, Department of Statistics; 1991.

Appendix

We list the six well known Christoffel Symbols as follows. For a detailed derivation, see Struik [7] or Grey [8].

$$\Gamma_{11}^{1} = \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)}, \qquad \Gamma_{12}^{2} = \frac{EG_u - FE_v}{2(EG - F^2)}$$

$$\Gamma_{11}^{2} = \frac{2EF_u - EE_v - FE_u}{2(EG - F^{2)})}, \qquad \Gamma_{22}^{1} = \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)}$$

$$\Gamma_{12}^{1} = \frac{GE_v - FG_u}{2(EG - F^2)}, \qquad \Gamma_{22}^{2} = \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}$$

In general, the solution of the geodesic equation depends upon a pair of partial differential equations as seen below:

$$\frac{d^2u}{ds^2} + \Gamma_{11}^{1}\left(\frac{du}{ds}\right)^2 + 2\,\Gamma_{12}^{1}\left(\frac{du}{ds}\frac{dv}{ds}\right) + \Gamma_{22}^{1}\left(\frac{dv}{ds}\right)^2 = 0$$

$$\frac{d^2v}{ds^2} + \Gamma_{11}^{2}\left(\frac{du}{ds}\right)^2 + 2\,\Gamma_{12}^{2}\left(\frac{du}{ds}\frac{dv}{ds}\right) + \Gamma_{22}^{2}\left(\frac{dv}{ds}\right)^2 = 0$$

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://sciencedomain.org/review-history/13529