Simple Method for Evaluating Singular Integrals

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Abstract

In this paper, we develop a method for evaluating one dimensional singular integrals (weakly, strongly, and hyper-singular) that converge in the sense of Cauchy principal value and Hadamard finite part integrals. A proof of convergence of this method is also provided.

Keywords

Singular Integral, Weakly Singular, Strongly Singular, Hyper Singular, Numerical Integration

1. Introduction

Many problems in engineering and science require evaluating singular integrals. The problem considered in this paper is of practical interest in many areas. For example, in electromagnetic and acoustic wave scattering, the boundary integral equations have singular kernels, see [1]-[6]. In fluid and solid mechanics, physicists and engineers face the same problem, see [7] [8]. Thus, the study of such integrals plays an important role in engineering and science.

The one dimensional singular integrals are defined in the literature as follows

$$\int_a^b \frac{u(t)}{(t-s)^p}\,dt, \quad s \in (a,b), \quad p > 0, \tag{1}$$

in which $u(t)$ is a continuous function. These integrals are classified by the order of singularity. If $p < 1$, the integral is called weakly singular. If $p = 1$, the integral is strongly singular. If $p > 1$, the integral is called hyper-singular, see [9]. In other words, an integral is called weakly singular if its value exists and continuous at the singularity. An integral is called strongly singular if both the integrand and integral are singular. An integral is called hyper-singular if the

kernel has a higher-order singularity than the dimension of the integral. For strongly singular integrals, they are often defined in terms of Cauchy principal value, see [10]. For hyper singular integrals, they are often interpreted as Hadamard finite part integrals, see [11].

There are many special methods developed to treat singular integral problems since numerical integration routines often lead to inaccurate solutions. For example, to deal with the singularities in surface integral equations, the method of moments regularizes the singular integrals by sourcing them analytically for specific observation point [12] [13]. Other methods include Gaussian quadrature method which has high-order of accuracy with a non-uniform mesh [14] [15], Newton-Cotes method which has low-order of accuracy with a uniform mesh [16] [17] [18], Guiggiani's method which extracts the singular parts of the integrand and treat them analytically [19], sigmoidal transformation which transforms the integrand to a periodic function [20] [21], and Duffy's transformation which cancels the singularity of type $\dfrac{1}{t}$ [22]. Most of these methods can be characterized in three categories: singularity subtraction, analytical transformation, and special purpose quadrature.

In this paper, we present an alternative approach for evaluating one dimensional singular integrals (weakly, strongly, and hyper-singular) which converge in the sense of Cauchy principal value and Hadamard finite part integrals. In addition, a proof of convergence of this method is outlined in Section 2 to serve as a theoretical basis for the method. In Section 3, the detailed implementations of our method are described for integrals over the standard interval $[-1,1]$ and arbitrary interval [a,b].

2. Approximation of Singular Integrals

Let $S = \int_D f(x) \mathrm{d}x$ be a singular integral. Without loss of generality, the singularity can be assumed to be at zero. We assume that the integral S has a finite value in the sense of principal value or Hadamard finite part integral, or in other words, this integral converges.

Let $D_\epsilon := D \setminus B_\epsilon(0)$, where $B_\epsilon(0) := \{x : |x| < \epsilon\}$, where $\epsilon > 0$. Then there exists numbers $\eta(n, \epsilon)$ and $a_j^{(\epsilon)}, 0 \le j \le n$, such that

$$\left| \int_{D_\epsilon} f(x) \mathrm{d}x - \sum_{j=0}^{n} a_j^{(\epsilon)} \int_{D_\epsilon} U_j(x) \mathrm{d}x \right| \le \eta(n, \epsilon) \to 0, \quad \text{as } n \to \infty, \tag{2}$$

where $U_j, 0 \le j \le n$, are Chebyshev polynomials of second kind

$$U_j(x) = \frac{\sin\left((j+1)\cos^{-1}(x)\right)}{\sin\left(\cos^{-1}(x)\right)}, \quad 0 \le j \le n. \tag{3}$$

Since the integral S converges, $\lim_{\epsilon \downarrow 0} \left| \int_{D_\epsilon} f(x) \mathrm{d}x - S \right| = 0$. Fix $\epsilon_0 > 0$, there exists a $\delta(\epsilon_0)$ such that

$$\left| \int_{D_{\epsilon_0}} f(x) \mathrm{d}x - S \right| < \delta(\epsilon_0) \quad \forall \epsilon \le \epsilon_0. \tag{4}$$

For a fixed $\epsilon = \epsilon_0$, there exists an n such that

$$\left| \int_{D_{\epsilon_0}} f(x) \mathrm{d}x - \sum_{j=0}^{n} a_j^{(\epsilon_0)} \int_{D_{\epsilon_0}} U_j(x) \mathrm{d}x \right| < \delta(\epsilon_0). \tag{5}$$

Thus,

$$\left| S - \sum_{j=0}^{n} a_j^{(\epsilon_0)} \int_{D_{\epsilon_0}} U_j(x) \mathrm{d}x \right| < 2\delta(\epsilon_0). \tag{6}$$

That means S can be approximated by $\sum_{j=0}^{n} a_j^{(\epsilon_0)} \int_{D_{\epsilon_0}} U_j(x) \mathrm{d}x$.

Let us summarize the result.

Theorem 1. Let $S = \int_D f(x) \mathrm{d}x$ be a singular integral that converges. Suppose x_0 is its singularity in D. Then, for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$, n, and $a_j^{(\epsilon)}, 0 \le j \le n$, such that

$$\left| S - \sum_{j=0}^{n} a_j^{(\epsilon)} \int_{D_\epsilon} U_j(x) \mathrm{d}x \right| < \delta(\epsilon), \quad D_\epsilon := D \setminus B_\epsilon(x_0). \tag{7}$$

3. Methods for Computing Singular Integrals

3.1. Integrals over $[-1,1]$

In this section, we present a method for evaluating singular integral of the type

$$S = \int_{-1}^{1} f(x) \mathrm{d}x. \tag{8}$$

For a more general integral, we will extend this method in Section 3.2.

From Section 2, we need to find the coefficient a_j such that

$$S \simeq \sum_{j=0}^{n-1} a_j \int_{-1}^{1} U_j(x) \mathrm{d}x, \tag{9}$$

or

$$f(x) \simeq \sum_{j=0}^{n-1} a_j U_j(x). \tag{10}$$

Since U_j are Chebyshev polynomials of second kind, they admit some nice properties

$$1. \ \int_{-1}^{1} U_j(x) \mathrm{d}x = \frac{2\sin^2\left(\dfrac{j\pi}{2}\right)}{j+1}, \tag{11}$$

$$2. \ \sum_{k=1}^{n} U_j(x_k) U_i(x_k) \left(1 - x_k^2\right) = \frac{n}{2} \delta_{ij}, \quad \forall i, j < n, \tag{12}$$

where $x_k, 1 \le k \le n$, are the roots of $U_n(x)$ in $[-1,1]$

$$x_k = \cos\left(\frac{k\pi}{n+1}\right), \quad 1 \le k \le n. \tag{13}$$

Now consider the following summation with $0 \leq i \leq n-1$

$$\sum_{k=1}^{n} f(x_k) U_i(x_k)(1-x_k^2) = \sum_{k=1}^{n}\sum_{j=0}^{n-1} a_j U_j(x_k) U_i(x_k)(1-x_k^2) \quad (14)$$

$$= \sum_{j=0}^{n-1} a_j \sum_{k=1}^{n} U_j(x_k) U_i(x_k)(1-x_k^2) \quad (15)$$

$$= \sum_{j=0}^{n-1} a_j \frac{n}{2}\delta_{ij} \quad (16)$$

$$= \frac{n}{2}a_i \quad (17)$$

Thus, the coefficient a_i can be computed by

$$a_i = \frac{2}{n}\sum_{k=1}^{n} f(x_k) U_i(x_k)(1-x_k^2), \quad 0 \leq i \leq n-1, \quad (18)$$

and

$$S \simeq \sum_{i=0}^{n-1} a_i \int_{-1}^{1} U_i(x)\,\mathrm{d}x = \sum_{i=0}^{n-1} a_i \frac{2\sin^2\left(\frac{i\pi}{2}\right)}{i+1}. \quad (19)$$

3.2. Integrals over [a,b]

In this section, we extend the method presented in Section 3.1 to singular integrals over arbitrary interval $[a,b]$

$$S = \int_a^b f(x)\,\mathrm{d}x. \quad (20)$$

Note that

$$\int_a^b U_i(x)\,\mathrm{d}x = \frac{b-a}{2}\int_{-1}^{1} U_i(y)\,\mathrm{d}y. \quad (21)$$

Therefore, we just need to recompute the coefficients a_i in this formula

$$S \simeq \sum_{i=0}^{n-1} a_i \int_a^b U_i(x)\,\mathrm{d}x = \frac{b-a}{2}\sum_{i=0}^{n-1} a_i \int_{-1}^{1} U_i(y)\,\mathrm{d}y. \quad (22)$$

Let $x \in [a,b]$ and $y \in [-1,1]$. Define

$$y_k := \cos\left(\frac{k\pi}{n+1}\right), \quad 1 \leq k \leq n, \quad (23)$$

the roots of $U_n(y)$ in $[-1,1]$. Then, using this change of variable formula to find the corresponding collocation points x_k in $[a,b]$

$$x_k = \frac{b-a}{2}y_k + \frac{b+a}{2}, \quad 1 \leq k \leq n. \quad (24)$$

Let

$$f(x_k) \simeq \sum_{j=0}^{n-1} a_j U_j(y_k), \quad 1 \leq k \leq n. \quad (25)$$

Then the coefficients a_i can be computed as follows. Consider the following summation with $0 \leq i \leq n-1$

$$\sum_{k=1}^{n} f\left(x_k\right) U_i\left(y_k\right)\left(1-y_k^2\right) = \sum_{k=1}^{n}\sum_{j=0}^{n-1} a_j U_j\left(y_k\right) U_i\left(y_k\right)\left(1-y_k^2\right) \tag{26}$$

$$= \sum_{j=0}^{n-1} a_j \sum_{k=1}^{n} U_j\left(y_k\right) U_i\left(y_k\right)\left(1-y_k^2\right) \tag{27}$$

$$= \sum_{j=0}^{n-1} a_j \frac{n}{2}\delta_{ij} \tag{28}$$

$$= \frac{n}{2}a_i \tag{29}$$

Thus

$$a_i = \frac{2}{n}\sum_{k=1}^{n} f\left(x_k\right) U_i\left(y_k\right)\left(1-y_k^2\right), \quad 0 \le i \le n-1. \tag{30}$$

Finally,

$$S = \int_a^b f\left(x\right)\mathrm{d}x \tag{31}$$

$$\simeq \sum_{i=0}^{n-1} a_i \int_a^b U_i\left(x\right)\mathrm{d}x \tag{32}$$

$$= \frac{b-a}{2}\sum_{i=0}^{n-1} a_i \int_{-1}^{1} U_i\left(y\right)\mathrm{d}y \tag{33}$$

$$= \frac{b-a}{2}\sum_{i=0}^{n-1} a_i \frac{2\sin^2\left(\dfrac{i\pi}{2}\right)}{i+1}. \tag{34}$$

4. Conclusion

In this paper, we investigated one dimensional singular integrals. A method is developed for evaluating weakly, strongly, and hyper singular integrals which converge in the sense of Cauchy principal value and Hadamard finite part integrals. The convergence of this method is proved and the detailed implementations are provided. One of the advantages of this method is that it is simple and relatively easy to implement. This method can serve as an alternative approach to other special methods in the literature.

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