American Journal of Computational Mathematics, 2017, 7, 435-443
http://www.scirp.org/journal/ajcm
ISSN Online: 2161-1211
ISSN Print: 2161-1203

Numerical Method for Solving Electromagnetic Scattering Problem by Many Small Impedance Bodies

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How to cite this paper: Tran, N.T. (2017) Numerical Method for Solving Electromagnetic Scattering Problem by Many Small Impedance Bodies. *American Journal of Computational Mathematics*, **7**, 435-443.
https://doi.org/10.4236/ajcm.2017.74031

Received: October 26, 2017
Accepted: December 11, 2017
Published: December 14, 2017

Abstract

In this paper, we study electromagnetic (EM) wave scattering problem by many small impedance bodies. A numerical method for solving this problem is presented. The problem is solved under the physical assumptions $ka \ll 1$, where a is the characteristic size of the bodies and k is the wave number. This problem is solved asymptotically and numerical experiments are provided to illustrate the idea of the method. Error estimate for the asymptotic solution is also discussed.

Keywords

Electromagnetic Scattering, Integral Equation, Boundary Impedance, Many-Body Scattering, EM Waves

1. Introduction

Electromagnetic scattering is the effect caused by EM waves such as light or radio waves hitting an object. The waves will then be scattered and the scattered field contains useful information about the object, see [1] [2] [3]. Electromagnetic scattering happens in many situations, for example, sun light scattered by atmosphere, radio waves scattered by buildings or planes, and so on. The study of EM wave scattering is of great interest and importance since it helps advance many different fields ranging from Medical Technology to Computer Engineering, Geophysics, Photonics, and Military Technology, see [2] [4]. Unfortunately, most wave equations cannot be solved analytically to get an exact solution. Therefore, numerical methods are sought to approximate the solution asymptotically. Unlike solving scalar wave scattering problem [5] [6], solving EM wave scattering is much more complicated and computationally expensive due to the vector nature of EM waves.

In [7] and [8], a theory for solving electromagnetic wave scattering problem by many small perfectly conducting and impedance bodies was developed. In [9] [10], numerical methods for solving EM wave scattering by one and many small perfectly conducting bodies are presented. In this paper, we focus on EM wave scattering by many small impedance bodies. A numerical method for solving this problem, based on the above theory, is described and tested. The problem is solved under the assumptions that the characteristic size a of the bodies is much smaller than the distance d between neighboring bodies, $d = O\left(a^{\frac{2-\kappa}{3}}\right)$ where $\kappa \in [0,1)$, and this distance d is much less than the wave length λ, $ka \ll 1$ where k is the wave number. The distribution of these small bodies is assumed to follow this law

$$\mathcal{N}(\Delta) = \frac{1}{a^{2-\kappa}} \int_\Delta N(x)\mathrm{d}x\left[1+o(1)\right], \quad a \to 0, \tag{1}$$

in which Δ is an arbitrary open subset of the domain Ω that contains all the small bodies, $\mathcal{N}(\Delta)$ is the number of the small bodies in Δ, and N is the distribution function of the bodies

$$N(x) \geq 0, \quad N(x) \in C(\Omega). \tag{2}$$

The boundary impedance of the bodies is of the form $\zeta = ha^{-\kappa}$, where h is a continuous function such that $\operatorname{Im} h \geq 0$, and $\kappa \in [0,1)$. The function h and constant κ can be chosen as desired.

To make the paper self-contained, the theory of EM wave scattering by one and many small impedance bodies is given in Sections 2 and 3. In Section 4, a numerical method for solving the EM scattering problem is presented. The solution of this problem is computed asymptotically and error analysis of the asymptotic solutions is also provided.

2. Electromagnetic Wave Scattering by One Small Impedance Body

Let D be a bounded domain of one small body, a be its radius, and S be its smooth boundary, $S \in C^{1,\gamma}, \gamma \in (0,1]$. Assume that the dielectric permittivity ϵ and magnetic permittivity μ are constants. Let E and H denote the electric field and magnetic field, respectively. E_0 is the incident field and v_E is the scattered field. The electromagnetic wave scattering by one small impedance body problem can be stated as follows

$$\nabla \times E = i\omega\mu H, \quad \text{in } D' := \mathbb{R}^3 \setminus D, \tag{3}$$

$$\nabla \times H = -i\omega\epsilon E, \quad \text{in } D', \tag{4}$$

$$\left[N,\left[E,N\right]\right] = \zeta\left[N,H\right], \quad \operatorname{Re}\zeta \geq 0, \tag{5}$$

$$E = E_0 + v_E, \tag{6}$$

$$E_0 = \mathcal{E}\mathrm{e}^{ik\alpha\cdot x}, \quad \mathcal{E}\cdot\alpha = 0, \quad \alpha \in S^2, \tag{7}$$

$$\frac{\partial v_E}{\partial r} - ikv_E = o\left(\frac{1}{r}\right), \quad r := |x| \to \infty, \tag{8}$$

where $\omega > 0$ is the frequency, $k = 2\pi/\lambda = \omega\sqrt{\epsilon\mu}$ is the wave number, $ka \ll 1$, λ is the wave length, ζ is the boundary impedance of the body, and α is a unit vector that indicates the direction of the incident wave E_0. This incident wave satisfies the relation $\nabla \cdot E_0 = 0$. The scattered field v_E satisfies the radiation condition (8). Here, N is the outward pointing unit normal to the surface S.

It is known from [8] that problem (3)-(8) has a unique solution and its solution is of the form

$$E(x) = E_0(x) + \nabla \times \int_S g(x,t) J(t)\mathrm{d}t, \quad g(x,t) := \frac{\mathrm{e}^{ik|x-t|}}{4\pi|x-t|}, \tag{9}$$

where E_0 is the incident plane wave defined in (7) and J is an unknown pseudovector. J is assumed to be tangential to S and can be found from the impedance boundary condition (5). Here E is a vector in $\mathbb{R}^3$ and $\nabla \times E$ is a pseudovector.

Once we have E, from (3) H can be found by the formula

$$H = \frac{\nabla \times E}{i\omega\mu}, \tag{10}$$

The asymptotic formula of E when the radius a of the body D tends to zero is

$$E(x) = E_0(x) + \left[\nabla_x g(x,x_1), Q\right], \tag{11}$$

where $|x - x_1| \gg a$, and the point x_1 is an arbitrary point inside the small body D, see [8]. So, instead of finding J to get E, we can just find one pseudovector Q

$$Q = \int_S J(t)\mathrm{d}t. \tag{12}$$

The analytical formula for Q is derived in [8] which can be summed up in the following theorem.

Theorem 1. *One has*

$$Q = -\frac{\zeta|S|}{i\omega\mu}\tau\nabla \times E_0 \tag{13}$$

where

$$\tau := I_3 - b, \quad b = \left(b_{jm}\right) := \frac{1}{|S|}\int_S N_j(s) N_m(s)\mathrm{d}s, \tag{14}$$

and $|S|$ is the surface area of S.

Here, $1 \le j,m \le 3$ correspond to x, y, and z coordinates in $\mathbb{R}^3$, I_3 is a 3 × 3 identity matrix, and $N_j, 1 \le j \le 3$ is the j-th component of the outer unit normal vector to the surface S.

3. Electromagnetic Wave Scattering by Many Small Impedance Bodies

Now, consider a domain Ω containing M small bodies D_m, $1 \le m \le M$, and

S_m are their corresponding smooth boundaries. Let $D := \bigcup_{m=1}^{M} D_m \subset \Omega$ and D' be the complement of D in $\mathbb{R}^3$. We assume that $S = \bigcup_{m=1}^{M} S_m \in C^{1,\gamma}, \gamma \in (0,1]$. We also assume that the dielectric permittivity ϵ and magnetic permittivity μ are constants. Let E and H denote the electric field and magnetic field, respectively. E_0 is the incident field and v is the scattered field. The electromagnetic wave scattering by many small impedance bodies problem involves solving the following system

$$\nabla \times E = i\omega\mu H, \quad \text{in } D' := \mathbb{R}^3 \setminus D, \quad D := \bigcup_{m=1}^{M} D_m, \tag{15}$$

$$\nabla \times H = -i\omega\epsilon E, \quad \text{in } D', \tag{16}$$

$$\left[N, [E, N]\right] = \zeta_m [N, H], \quad \text{on } S_m, \ 1 \le m \le M, \tag{17}$$

$$E = E_0 + v, \tag{18}$$

$$E_0 = \mathcal{E} e^{ik\alpha \cdot x}, \quad \mathcal{E} \cdot \alpha = 0, \quad \alpha \in S^2. \tag{19}$$

where v satisfies the radiation condition (8), $\omega > 0$ is the frequency, $k = 2\pi/\lambda$ is the wave number, $ka \ll 1$, $a := \frac{1}{2}\max_m \operatorname{diam} D_m$, α is a unit vector that indicates the direction of the incident wave E_0, and ζ_m is the boundary impedance of the body D_m. These ζ_m's are given by the following formula

$$\zeta_m = \frac{h(x_m)}{a^\kappa}, \quad 0 \le \kappa < 1, \quad x_m \in D_m, \quad 1 \le m \le M, \tag{20}$$

where $h(x)$ is a continuous function in a bounded domain Ω,

$$\operatorname{Re} h(x) \ge 0, \quad \operatorname{Im} \varepsilon = \frac{\sigma}{\omega} \ge 0, \tag{21}$$

$$\varepsilon = \varepsilon_0, \quad \mu = \mu_0 \quad \text{in } \Omega' := \mathbb{R}^3 \setminus \Omega. \tag{22}$$

The distribution of small bodies D_m, $1 \le m \le M$, in Ω satisfies the following assumption

$$\mathcal{N}(\Delta) = \frac{1}{a^{2-\kappa}} \int_\Delta N(x) \mathrm{d}x \left[1 + o(1)\right], \quad a \to 0, \tag{23}$$

where $\mathcal{N}(\Delta)$ is the number of small bodies in Δ, Δ is an arbitrary open subset of Ω,

$$N(x) \ge 0, \quad N(x) \in C(\Omega) \tag{24}$$

and $\kappa \in [0,1)$ is the parameter from (20).

From (15) and (16) we have

$$\nabla \times \nabla \times E = k^2 E, \quad k^2 = \omega^2 \epsilon \mu, \tag{25}$$

if $\mu = \text{const}$. Once we have E, then H can be found from this relation

$$H = \frac{\nabla \times E}{i\omega\mu}. \tag{26}$$

From (26) and (25), one can get (16). Thus, we need to find only E which satisfies the boundary condition (17). It was proved in [8] that under the assumptions (21), the problem (15)-(18) has a unique solution and its solution is of the form

$$E(x)=E_0(x)+\sum_{m=1}^{M}\nabla\times\int_{S_m}g(x,t)J_m(t)\mathrm{d}t. \tag{27}$$

where

$$Q_m:=\int_{S_m}J_m(t)\mathrm{d}t. \tag{28}$$

When $a\to 0$, the asymptotic solution for the electric field is given by

$$E(x)=E_0(x)+\sum_{m=1}^{M}\left[\nabla g(x,x_m),Q_m\right]. \tag{29}$$

Therefore, instead of finding $J_m(t), 1\le m\le M$, we can just find Q_m. The analytic formula for Q_m is derived in [8] by using formula (13) and replacing E_0 in this formula by the effective field E_{em} acting on the m-th body

$$Q_m=-\frac{\zeta_m|S_m|}{i\omega\mu}\tau_m\nabla\times E_{em},\quad 1\le m\le M. \tag{30}$$

The effective field acting on the m-th body is defined as

$$E_e(x_m)=E_0(x_m)+\sum_{j\neq m}^{M}\left[\nabla g(x,x_j),Q_j\right]\Big|_{x=x_m}, \tag{31}$$

and $E_{em}:=E_e(x_m)$, x_m is a point in D_m. When $a\to 0$, the effective field $E_e(x)$ is asymptotically equal to the field $E(x)$ in (29) as proved in [8].

From (20), (30), and (31), one gets

$$E_{em}=E_{0m}-\frac{ca^{2-\kappa}}{i\omega\mu}\sum_{j\neq m}^{M}\left[\nabla g(x,x_j)\Big|_{x=x_m},\tau\nabla\times E_{ej}\right]h_j, \tag{32}$$

where c is a positive constant depending on the shape of the body S_m, $|S_m|=ca^2$, $\tau_m=\tau:=I_3-b$, and

$$b=(b_{jn}):=\frac{1}{|S_m|}\int_{S_m}N_j(s)N_n(s)\mathrm{d}s. \tag{33}$$

Here, $1\le j,n\le 3$ correspond to x, y, and z coordinates in $\mathbb{R}^3$, I_3 is a 3 × 3 identity matrix, and N_j, $1\le j\le 3$, is the *j*-th component of the outer unit normal vector to the surface S_m.

4. Numerical Method for Solving EM Scattering Problem by Many Small Impedance Bodies

Our goal is to find E_{em} in (32). Take curl of (32), set $x=x_j$, and let $A_m:=\nabla\times E_{em}$, we have

$$A_j=A_{0j}-\frac{ca^{2-\kappa}}{i\omega\mu}\sum_{m\neq j}^{M}k^2g(x_j,x_m)\tau A_mh_m+(\tau A_m\cdot\nabla_x)\nabla g(x,x_m)\Big|_{x=x_j}h_m, \tag{34}$$

where $1\le j\le M$, see [8]. Solving this linear system gives us the curl of E_{em}, for $1\le m\le M$.

This linear system can be solved directly using Gaussian elimination method.

Then E can be computed as follows

$$E_{em}=E_{0m}-\frac{ca^{2-\kappa}}{i\omega\mu}\sum_{j\neq m}^{M}\left[\nabla g\left(x,x_j\right)\Big|_{x=x_m},\tau A_j\right]h_j. \quad (35)$$

Matrix τ is of size 3x3 and can be computed as follows

$$\tau:=I_3-b=\frac{2}{3}I_3,\quad b=\left(b_{jm}\right):=\frac{1}{|S|}\int_S N_j\left(s\right)N_m\left(s\right)\mathrm{d}s. \quad (36)$$

Let $A_i:=\left(X_i,Y_i,Z_i\right)$ then one can rewrite (34) as

$$F_x\left(i\right)=X_i+\sum_{j\neq i}^{M}a_{ij}X_j+\sum_{j\neq i}^{M}b_{ij}Y_j+\sum_{j\neq i}^{M}c_{ij}Z_j, \quad (37)$$

$$F_y\left(i\right)=Y_i+\sum_{j\neq i}^{M}a'_{ij}X_j+\sum_{j\neq i}^{M}b'_{ij}Y_j+\sum_{j\neq i}^{M}c'_{ij}Z_j, \quad (38)$$

$$F_z\left(i\right)=Z_i+\sum_{j\neq i}^{M}a''_{ij}X_j+\sum_{j\neq i}^{M}b''_{ij}Y_j+\sum_{j\neq i}^{M}c''_{ij}Z_j, \quad (39)$$

where by the subscripts x,y,z the corresponding coordinates are denoted, e.g. $F\left(i\right)=\left(F_x,F_y,F_z\right)\left(i\right)$, $F\left(i\right):=A_{0i}$, and

$$a_{ij}:=\left[k^2g\left(i,j\right)+\partial x\nabla g\left(i,j\right)_x\right]D_j, \quad (40)$$

$$b_{ij}:=\partial y\nabla g\left(i,j\right)_x D_j, \quad (41)$$

$$c_{ij}:=\partial z\nabla g\left(i,j\right)_x D_j, \quad (42)$$

$$a'_{ij}:=\partial x\nabla g\left(i,j\right)_y D_j, \quad (43)$$

$$b'_{ij}:=\left[k^2g\left(i,j\right)+\partial y\nabla g\left(i,j\right)_y\right]D_j, \quad (44)$$

$$c'_{ij}:=\partial z\nabla g\left(i,j\right)_y D_j, \quad (45)$$

$$a''_{ij}:=\partial x\nabla g\left(i,j\right)_z D_j, \quad (46)$$

$$b''_{ij}:=\partial y\nabla g\left(i,j\right)_z D_j, \quad (47)$$

$$c_{ij}:=\left[k^2g\left(i,j\right)+\partial z\nabla g\left(i,j\right)_z\right]D_j, \quad (48)$$

in which $D_j:=\frac{2}{3}\frac{ca^{2-\kappa}}{i\omega\mu}h_j$.

5. Error Analysis

The error of the method presented in Section 4 can be estimated as follows. From the solution E of the electromagnetic scattering problem by many small bodies given in (27)

$$E\left(x\right)=E_0\left(x\right)+\sum_{m=1}^{M}\nabla\times\int_{S_m}g\left(x,t\right)J_m\left(t\right)\mathrm{d}t, \quad (49)$$

we can rewrite it as

$$E\left(x\right)=E_0\left(x\right)+\sum_{m=1}^{M}\left[\nabla g\left(x,x_m\right),Q_m\right]+\sum_{m=1}^{M}\nabla\times\int_{S_m}\left[g\left(x,t\right)-g\left(x,x_m\right)\right]J_m\left(t\right)\mathrm{d}t. \quad (50)$$

Comparing this with the asymptotic formula for E when $a \to 0$ given in (29)

$$E(x) = E_0(x) + \sum_{m=1}^{M} \left[\nabla g(x, x_m), Q_m \right], \tag{51}$$

we have the error of this asymptotic formula is

$$\text{Error} = \left| \sum_{m=1}^{M} \nabla \times \int_{S_m} \left[g(x,t) - g(x, x_m) \right] J_m(t) \mathrm{d}t \right| \tag{52}$$

$$\sim \frac{1}{4\pi} \left(\frac{ak^2}{d} + \frac{ak}{d^2} + \frac{a}{d^3} \right) \sum_{m=1}^{M} |Q_m|, \tag{53}$$

where $d = \min_m |x - x_m|$ and

$$Q_m := \int_{S_m} J_m(t) \mathrm{d}t \simeq -\frac{\zeta_m |S_m|}{i\omega\mu} \tau_m \nabla \times E_{em} = -\frac{\zeta_m |S_m|}{i\omega\mu} \tau A_m, \quad a \to 0, \tag{54}$$

because

$$\left| \nabla \left[g(x,t) - g(x, x_m) \right] \right| = O\left(\frac{ak^2}{d} + \frac{ak}{d^2} + \frac{a}{d^3} \right), \quad a = \max_m |t - x_m|. \tag{55}$$

Thus, to reduce the error, one needs to reduce the quantity ka.

6. Experiments

To illustrate the idea of the method described in 4, consider a domain Ω as a unit cube placed in the first octant such that the origin is one of its vertex. This domain Ω contains M small bodies. The small bodies are particles which are distributed uniformly in the unit cube. The following physical parameters are used to solve the problem

- Speed of wave, $c = 3.0E + 10$ cm/sec.
- Frequency, $\omega = 1.0E + 14$ Hz.
- Wave number, $k = 2.094395e + 04\ \text{cm}^{-1}$.
- Constant $\kappa = 0.9$.
- Volume of the domain Ω that contains all the particles, $|\Omega| = 1\ \text{cm}^3$.
- Direction of plane wave, $\alpha = (1,0,0)$.
- Vector $\mathcal{E} = (0,1,0)$.
- Function $N(x) = Ma^{2-\kappa}/|\Omega|$ where M is the total number of particles and a is the radius of one particle.
- Function $h(x) = 1$.
- Function $\mu(x) = 1$.
- The distance between two neighboring particles, $d = 1/(b-1)$ cm, where b is the number of particles on a side of the cube.
- Vector A_0: $A_{0m} := A_0(x_m) = \nabla \times E_0(x)\big|_{x=x_m} = \nabla \times \mathcal{E} e^{ik\alpha \cdot x}\big|_{x=x_m}$.

The radius a of the particles is chosen variously so that it satisfies and dissatisfies the assumption $ka \ll 1$. The numerical solution to EM wave scattering problem by many small impedance bodies is computed for $M = 125$ and 1000 particles.

Table 1 and Figure 1 show the results of solving the electromagnetic wave scattering problem with $M = 125$ particles, the distance between neighboring particles is $d = 2.50E - 01$ cm, and with different radius *a* of particles. When the radius of particles decreases from $1.0E - 4$ cm to $1.0E - 6$ cm, the error of the asymptotic solution decreases rapidly from about $7.07E - 08$ to $4.46E - 12$. Note that when $a = 1.0E - 4$ cm, $ka > 1$, which does not satisfy the assumption $ka \ll 1$. When $ka < 1$, the error of the solution is always less than $7.0E - 08$.

Table 2 and Figure 2 show the results of solving the problem with $M = 1000$ particles, when the distance between neighboring particles is $d = 1.25E - 01$ cm, and with different radius *a*. From Table 2, one can see that the error of the asymptotic solution is also very small when $ka \ll 1$, less than $1.13E - 06$, but greater than the previous case when $M = 125$ for corresponding *a*. This time, the error is also decreasing rather quickly when the radius of the particles decreases from $1.0E - 4$ cm to $1.0E - 6$ cm. The small error when $ka \ll 1$ guarantees that the asymptotic formula (31) for the solution E is well applicable under this assumption.

Table 1. Error of the asymptotic solution E when $M = 125$ and $d = 2.50E - 01$ cm.

$M = 125$, $d = 2.50E - 01$					
a	1.00E−04	5.00E−05	1.00E−05	5.00E−06	1.00E−06
Norm of *E*	1.12E+01	1.12E+01	1.12E+01	1.12E+01	1.12E+01
Error of *E*	7.07E−08	1.65E−08	5.61E−10	1.31E−10	4.46E−12

Table 2. Error of the asymptotic solution E when $M = 1000$ and $d = 1.25E - 01$ cm.

$M = 1000$, $d = 1.25E - 01$					
a	1.00E−04	5.00E−05	1.00E−05	5.00E−06	1.00E−06
Norm of *E*	3.16E+01	3.16E+01	3.16E+01	3.16E+01	3.16E+01
Error of *E*	1.13E−06	2.63E−07	8.96E−09	2.09E−09	7.11E−11

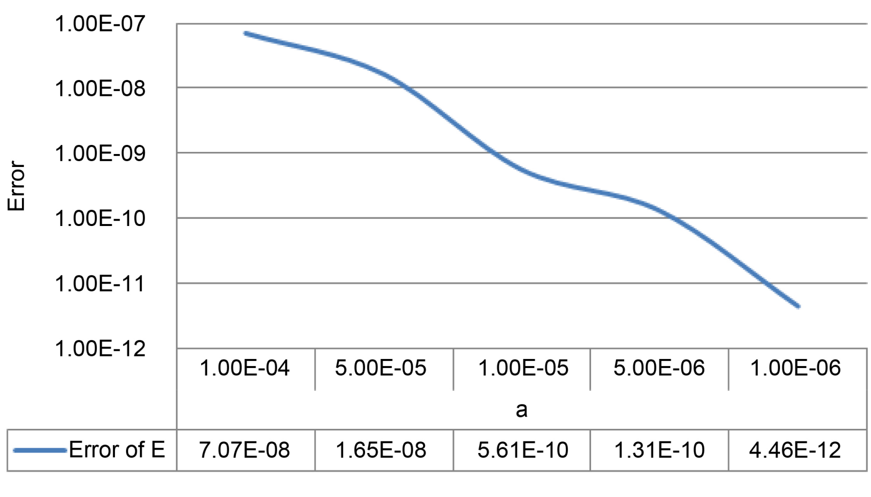
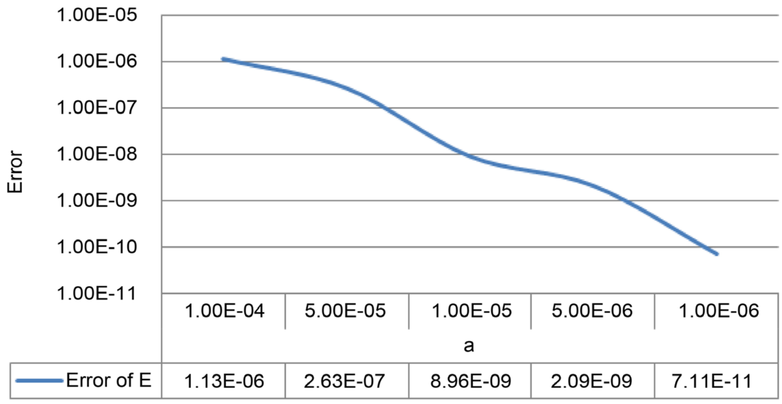


Figure 1. Error of the asymptotic solution E when $M = 125$ and $d = 2.50E - 01$ cm.



	1.00E-04	5.00E-05	1.00E-05	5.00E-06	1.00E-06
Error of E	1.13E-06	2.63E-07	8.96E-09	2.09E-09	7.11E-11

Figure 2. Error of the asymptotic solution E when $M = 1000$ and $d = 1.25E-01$ cm.

7. Conclusion

In this paper, we present a numerical method for solving EM wave scattering problem by many small impedance bodies. For illustration, numerical experiments are also provided. The solution to the EM wave scattering problem can be computed numerically and asymptotically using the described method, and the result is highly accurate if the assumption $ka \ll 1$ is satisfied.

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