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Hierarchical Stackelberg Control for a Two-Stroke Linear System in Population Dynamics

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Authors' contributions

This work was carried out in collaboration between all authors. Author FN designed the study, performed the statistical analysis, wrote the protocol, and wrote the first draft of the manuscript. Authors MK and MS managed the analyses of the study. Author MS managed the literature searches. All authors read and approved the final manuscript.

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Abstract

We study a unique hierarchical control problem for a two-stoke linear system, adjoint to an age and space dependent population dynamics problem. Using Stackelberg's method, we introduce two levels of control: a boundary control to achieve optimal flow regulation and a distributed control for null controllability. Our approach employs Carleman inequalities to address non-homogeneous Dirichlet boundary conditions, leading to new insights in controlling invasive species populations. These results highlight the applicability of hierarchical controls in ecological systems, providing a robust framework for future studies in control theory and population dynamics.

Keywords: Hierarchical stackelberg; stroke linear system; leadership model.

2010 Mathematics Subject Classification: 35K05, 49J20, 93C05, 93B05.

1 Introduction

We consider a population with age dependence and spatial structure, and we assume that the population lives in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}\setminus \{0\}$ with boundary $\Gamma := \Gamma_0 \cup \Gamma_1$ of class C^2 verifying $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 \neq \emptyset$. Let $q = q(t, a, x)$ be the distribution of individuals of age $a \in [0, A]$ at time $t \in [0, T]$ and location $x \in \Omega$. Let also $A \geq 0$ be the life expectancy of an individual and the final time $T \geq 0$. Let $\mathcal{O}$ and ω be a nonempty subsets of Ω. We set $Q = (0, T) \times (0, A) \times \Omega$, $\Omega^T = (0, T) \times \Omega$, $\Omega^A = (0, A) \times \Omega$, $\mathcal{O}^{TA} = (0, T) \times (0, A) \times \mathcal{O}$, $\omega^{TA} = (0, T)\times(0, A)\times\omega$, $\Sigma = (0, T)\times(0, A)\times\Gamma$ and $\Sigma_1 = (0, T)\times(0, A)\times\Gamma_1$. We denote by $\mu = \mu(t, a, x) \geq 0$, the natural death rate of individuals of age a at time t and location x. Then, we consider the following linear system:

$$\begin{cases} -\dfrac{\partial q}{\partial t} - \dfrac{\partial q}{\partial a} - \Delta q + \mu q &= h_0\chi_{\mathcal{O}} + k\chi_\omega \quad \text{in} \quad Q, \\ q(t, a, x) &= v\chi_{\Gamma_0} \quad \text{on} \quad \Sigma, \\ q(T, a, x) &= 0 \quad \text{in} \quad \Omega^A, \\ q(t, A, x) &= 0 \quad \text{in} \quad \Omega^T, \end{cases} \tag{1.1}$$

where the controls v and k belong respectively to $L^2(\omega_2^{TA})$ and $L^2(\Sigma_0)$, $\chi_{\mathbb{X}}$ denotes the characteristic function on the open set $\mathbb{X}$. The function h_0 is know and represents the supply of individuals. We make the following assumptions

$$\begin{cases} \mu(t, a, x) = \mu_0(a) + \mu_1(t, a, x) \text{ in } Q, \\ \mu_1 \in L^\infty(Q);\ \mu_1(t, a, x) \geq 0 \text{ for } (t, a, x) \text{ in } Q, \\ \mu_0 > 0,\ \mu_0 \in L^1_{loc}(0, A),\ \lim_{a\to A} \int_0^a \mu_0(s)ds = +\infty. \end{cases} \tag{1.2}$$

Under the above assumptions on the data, it is well known that system (1.1) has a unique solution $q(v, k) = q(t, a, x; v, k) \in H^{2,1}(Q)$ where from now on

$$H^{r,s}((0, T) \times (0, A) \times \mathbb{X}) = L^2((0, T) \times (0, A); H^r(\mathbb{X})) \cap H^s((0, T) \times (0, A); L^2(\mathbb{X})).$$

Moreover, there exist a positive constante $C = C(T)$ such that

$$\|q\|_{H^{2,1}(Q)} \leq C \left(\|h_0\|_{L^2(\mathcal{O}^{TA})} + \|v\|_{L^2(\Sigma_0)} + \|k\|_{L^2(\omega^{TA})}\right), \tag{1.3}$$

and it follows from the continuity of the trace that,

$$\left\|\frac{\partial q}{\partial \nu}\right\|^2_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} \leq C\left(\|h_0\|_{L^2(\mathcal{O}^{TA})} + \|v\|_{L^2(\Sigma_0)} + \|k\|_{L^2(\omega^{TA})}\right). \tag{1.4}$$

Remark 1.1. $H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ is a Sobolev space, where the index $\frac{1}{2}$ represents the spatial regularity on $\mathbb{X}$ and the index $\frac{1}{4}$ represents the regularity with respect to time t and age a on the Σ boundary.

The system (1.1) can describe the adjoint of a dynamics problem of an invasive species considered as a threat in a domain that can be materialised as a cornfield, where the supply of the invasive species is known, which translates into the function h_0. So, we want to drive the distribution of the invasive species to zero at the initial time with appropriate control acting on a sub-domain of the cornfield, trying meanwhile to keep the flux of the invasive species to zero with another control acting in another of boundary of the cornfield during the time interval $(0, T)$. We thus consider the following problems.

Problem 1. ***(Optimal control problem)*** *Let Ω be a bounded open set of $\mathbb{R}^n$, $n \in \mathbb{N}^*$ with bounary $\Gamma := \Gamma_0 \cup \Gamma_1$ of class C^2 verifying $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 \neq \emptyset$ and ω be a non-empty subset of Ω. Given $k \in L^2(\omega^{TA})$, find the control $u := u(k) \in L^2(\Sigma_0)$ such that*

$$J(u) = \underset{v \in L^2(\Sigma_0)}{inf} J(v), \tag{1.5}$$

where

$$J(v) = \left\|\frac{\partial q}{\partial \nu}(v,k)\right\|^2_{L^2(\Sigma_1)} + \alpha\|v\|^2_{L^2(\Sigma_0)}, \tag{1.6}$$

with $\alpha > 0$, ν is the unit exterior normal vector of Γ, $\dfrac{\partial q}{\partial \nu}$ is the normal derivated of q with respect to ν and $q(v, k)$ is solution of system (1.1).

Problem 2. ***(Null controllability problem)*** *Let ω be a non empty subset of Ω. Let also $u(k)$ be the optimal control obtain in the Problem 1. Then, find a control $k \in L^2(\omega^{TA})$ such that if $q = q(t, a, x; u(k), k)$ is solution of the system* (1.1)*, then*

$$q(0) = q(0, a, x; u(k), k) = 0 \quad \text{in} \quad \Omega^A. \tag{1.7}$$

Remark 1.2. Note that from (1.4), the cost function defined by (1.6) is well defined.

The Stackelberg leadership model is a multiple-objective optimization approach proposed by H. Von Stackelberg in [1]. This model involves two companies (controls) which compete on the market of the same product. The first(leader) to act must integrate the reaction of the other firms (followers) in the choices it makes in the amount of product that it decides to put on the market. From a mathematical point of view, more specifically in the context of partial differential equations the Stackelberg strategy was introduced by J-L. Lions in [2, 3], the author used respectively the Stackelberg strategy for the parabolic and wave equation and subjected to two controls. M.Mercan et al. in [4] use the hierachical control for the adjoint of a dynamics problem with costraint on the state. The authors used the hierarchic control which combine the controllability problem with robustness. The follower was responsible for a null controllability problem, while the leader addressed an optimal control problem. G. Mophou et al. studied in [5] the hierarchical control for a population dynamics model with the distribution of newborns as unknown. The leader was in charge of a null controllability problem while the follower solved an optimal control problem in presence of the missing data. Recently in 2022 [6], the authors revisits the hierachical control for a degenerate parabolic equation with missing data. The authors use of the low-regret control to solve the associated optimal control problem. For more literature on stackelberg control, we refer the reader to [7, 8, 9, 10, 11, 12, 13, 4, 14, 15, 16, 17, 18, 19, 20]. The authors used the hierarchic control which combine the controllability problem with robustness.

In the literature on Stackelberg strategy, most controls are distributed. For what concerns boundary controls, we can reference the works in [21, 22, 23, 24]. In [24], the authors present a Stackelberg-Nash strategy for the heat equation combining the concepts of controllability with robustness: the main control (the leader) is in charge of a null-controllability objective, while a secondary control (the follower) solves a robust control problem. First, the authors consider the case with a boundary follower and a distributed leader, and secondly, the case with a distributed follower and a boundary leader. Finally, they examine the possibility and limitations of placing all the involved controls on the boundary. In [21], the authors revisit a Stackelberg-Nash strategy in [24] and consider the case where all controls are boundary.

This work revisits the Stackelberg strategy used in [22, 24] and applies it to an adjoint population dynamics problem. The leader, responsible for a null controllability problem, uses a distributed control, while the follower, solving an optimal control problem, uses boundary control which consist in bringing the flow of two strok linear system to zero. We obtain specific adjoint systems with non-homogeneous Dirichlet-type conditions. to solve the null controllability problem, we use a Carleman-type observability inequality associated with a system of non-homogeneous Dirichlet boundary conditions. Additionally, we apply the Poincaré inequality and the continuity property of the trace operator to relate the boundary integral to a volume integral on Ω.

More precisely, we prove the following results.

2 Main Results

The result obtain when solving optimal control problem (Problem 1) is as follows

Theorem 2.1. *Let Ω be a bounded open set of $\mathbb{R}^n$, $n \in \mathbb{N}^*$ with boundary $\Gamma := \Gamma_0 \cup \Gamma_1$ of class C^2 verifying $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 \neq \emptyset$ and ω, $\mathcal{O}$ be a non-empty subset of Ω. Let also $k \in L^2(\omega^{TA})$. Then, there exist $p \in L^2(Q)$ such that the optimal control problem* (1.5) *has a unique solution $u \in L^2(\Sigma_0)$ which is characterized by the following optimality system:*

$$\left\{\begin{array}{rcll} -\dfrac{\partial q}{\partial t} - \dfrac{\partial q}{\partial a} - \Delta q + \mu q & = & h_0\chi_{\mathcal{O}} + k\chi_\omega & \text{in} \quad Q, \\ q(t,a,x) & = & u\chi_{\Gamma 0} & \text{on} \quad \Sigma, \\ q(T,a,x) & = & 0 & \text{in} \quad \Omega^A, \\ q(t,A,x) & = & 0 & \text{in} \quad \Omega^T, \end{array}\right. \tag{2.1}$$

$$\left\{\begin{array}{rcll} \dfrac{\partial p}{\partial t} + \dfrac{\partial p}{\partial a} - \Delta p + \mu p & = & 0 & \text{in} \quad Q, \\ p(t,a,x) & = & \dfrac{\partial q}{\partial \nu}\chi_{\Gamma 1} & \text{on} \quad \Sigma, \\ p(0,a,x) & = & 0 & \text{in} \quad \Omega^A, \\ p(t,0,x) & = & 0 & \text{in} \quad \Omega^T, \end{array}\right. \tag{2.2}$$

and

$$u = -\frac{\frac{\partial p}{\partial \nu}}{\alpha} \quad \text{on} \quad \Sigma_0. \tag{2.3}$$

Moreover there exists a positif constant $C = C(\alpha, T)$ such that

$$\|u\|_{L^2(\Sigma_0)} \leq C\left(\|h_0\|_{L^2(\mathcal{O}^{TA})} + \|k\|_{L^2(\omega^{TA})}\right). \tag{2.4}$$

The result obtain when solving the null controlability problem (Problem2) is as follows

Theorem 2.2. *Assume that the assumptions of Theorem 2.1 hold. Assume also that $\mathcal{O}$ and ω be nonempty subsets of Ω are such that $\mathcal{O} \subset \omega$. Then there exists a positive weight function $\theta \in L^\infty(Q)$ to be define later by* (3.27) *such that for any $h_0 \in L^2(\mathcal{O}^{TA})$ with $\theta h_0 \in L^2(\mathcal{O}^{TA})$. There exists a unique control $\hat{k} \in L^2(\omega^{TA})$ such that $(\hat{k}, q, p)$ is a solution of the null controllability problem* (2.1)-(2.3). *Moreover*

$$\hat{k} = \hat{\rho} \quad \text{in} \quad \omega^{TA}, \tag{2.5}$$

where $\hat{\rho}$ satisfies

$$\begin{cases} \dfrac{\partial \hat{\rho}}{\partial t} + \dfrac{\partial \hat{\rho}}{\partial a} - \Delta\hat{\rho} + \mu\hat{\rho} &= 0 & \text{in} & Q, \\ \hat{\rho}(t,a,x) &= \dfrac{\partial \hat{\varpi}}{\partial \nu}\chi_{\Gamma 1} & \text{on} & \Sigma, \\ \hat{\rho}(t,0,x) &= 0 & \text{in} & \Omega^T, \end{cases} \tag{2.6}$$

where $\hat{\varpi}$ solution of

$$\begin{cases} -\dfrac{\partial \hat{\varpi}}{\partial t} - \dfrac{\partial \hat{\varpi}}{\partial a} - \Delta\hat{\varpi} + \mu\hat{\varpi} &= 0 & \text{in} & Q, \\ \hat{\varpi}(t,a,x) &= -\dfrac{\frac{\partial \hat{\rho}}{\partial \nu}}{\alpha}\chi_{\Gamma 0} & \text{on} & \Sigma, \\ \hat{\varpi}(T,a,x) &= 0 & \text{in} & \Omega^A, \\ \hat{\varpi}(t,A,x) &= 0 & \text{in} & \Omega^T. \end{cases} \tag{2.7}$$

In addition, there exists a positif constant $C = C(T)$ such that

$$\|\hat{k}\|_{L^2(\omega^{TA})} \leq \sqrt{C}\|\theta h_0\|_{L^2(\mathcal{O}^{TA})}. \tag{2.8}$$

The rest of the work will be organized as follows. In Section 3, we proov the Main results. In Section 4, we provide a discussion, which is optionally available, summarizing the implications of the findings. Finally, in Section 5 we conclude.

3 Proof of Main Results

In this section, we are interested in the resolution of optimal control problem and null controllability problem

3.1 Proof of optimal control problem

we prove the existence and uniqueness of the optimal control solution, then we characterize it, and finally, we provide an estimation of the optimal control

Proposition 3.1. *For any $k \in L^2(\omega^{TA})$, there exists a unique optimal control $u := u(k) \in L^2(\Sigma_0)$ such that* (1.5) *holds true.*

Proof. It obtain that the fonctionnal J is coercive and strictly convex on $L^2(\Sigma_0)$. The fonctionnal J is continuous on $L^2(\Sigma_0)$, Indeed we decompose the solution $q(v;k) := q$ of the system (1.1) as follows:

$$q(v;k) = z(v) + \tau(k), \tag{3.1}$$

such that $z(v) := z$ and $\tau(k) := \tau$ are respectively solutions of the following systems:

$$\begin{cases} -\dfrac{\partial z}{\partial t} - \dfrac{\partial z}{\partial a} - \Delta z + \mu z &= 0 & \text{in} & Q, \\ z(t,a,x) &= v\chi_{\Gamma 0} & \text{on} & \Sigma, \\ z(T,a,x) &= 0 & \text{in} & \Omega^A, \\ z(t,A,x) &= 0 & \text{in} & \Omega^T, \end{cases} \tag{3.2}$$

and

$$\begin{cases} -\dfrac{\partial \tau}{\partial t} - \dfrac{\partial \tau}{\partial a} - \Delta \tau + \mu \tau &= h_0 \chi_{\mathcal{O}} + k \chi_\omega & \text{in} & Q, \\ \tau(t,a,x) &= 0 & \text{on} & \Sigma, \\ \tau(T,a,x) &= 0 & \text{in} & \Omega^A, \\ \tau(t,A,x) &= 0 & \text{in} & \Omega^T, \end{cases} \tag{3.3}$$

Under the assumptions on the data, the systems (3.2) and (3.3) admit a unique solution in $H^{2,1}(Q)$. Moreover we have the existence of a positive constant $C = C(T)$ such that the following estimates are verified:

$$\|z\|_{H^{2,1}(Q)} \leq C\|v\|_{L^2(\Sigma_0)}, \tag{3.4}$$

and

$$\|\tau\|_{H^{2,1}(Q)} \leq C\|h_0\chi_{\mathcal{O}} + k\chi_\omega\|_{L^2(Q)}. \tag{3.5}$$

Using the decomposition of q given by (3.1). We have that the cost fonction defined by (1.6) can be rewrittent as

$$J(v) = \mathcal{P}(v) + \mathcal{L}(v), \ \forall \ v \in L^2(\Sigma_0),$$

with

$$\mathcal{P}(v) = \left\| \frac{\partial z(v)}{\partial \nu} \right\|^2_{L^2(\Sigma_1)} + \alpha\|v\|^2_{L^2(\Sigma_0)},$$

and

$$\mathcal{L}(v) = 2\langle \frac{\partial z(v)}{\partial \nu}; \frac{\partial \tau(k)}{\partial \nu} \rangle_{L^2(\Sigma_1)} + \left\| \frac{\partial \tau(k)}{\partial \nu} \right\|^2_{L^2(\Sigma_1)}.$$

It follows from the continuity of the trace and Scalar product , we have

$$v \mapsto \mathcal{P}(v),$$

and

$$v \mapsto \mathcal{L}(v),$$

are continuous on $L^2(\Sigma_0)$. Hence $v \mapsto J(v)$ is continuous.

It then follows there exists a unique optimal control $u = u(k) \in L^2(\Sigma_0)$ such that (1.5) holds. □

In order to characterize the optimal control u, we express the Euler Lagrange optimality conditions:

$$\lim_{\lambda \to 0} \frac{J(u + \lambda v) - J(u)}{\lambda} = 0, \qquad \forall \quad v \in L^2(\Sigma_0). \tag{3.6}$$

Using the structure of the functional J given by (1.5), we have:

$$\begin{aligned}
J(u+\lambda v) &= \left\| \frac{\partial q}{\partial \nu}(u+\lambda v) \right\|^2_{L^2(\Sigma_1)} + \alpha \|u+\lambda v\|^2_{L^2(\Sigma_0)} \\
&= \left\| \frac{\partial q}{\partial \nu}(u+\lambda v) - \frac{\partial q}{\partial \nu}(u) + \frac{\partial q}{\partial \nu}(u) \right\|^2_{L^2(\Sigma_1)} + \alpha \|u+\lambda v\|^2_{L^2(\Sigma_0)} \\
&= \left\| \frac{\partial q}{\partial \nu}(u+\lambda v) - \frac{\partial q}{\partial \nu}(u) \right\|^2_{L^2(\Sigma_1)} + \left\| \frac{\partial q}{\partial \nu}(u) \right\|^2_{L^2(\Sigma_1)} + \alpha \|u\|^2_{L^2(\Sigma_0)} \\
&+ \alpha \lambda^2 \|v\|^2_{L^2(\Sigma_0)} + 2\int_{\Sigma_1} \left(\frac{\partial q}{\partial \nu}(u+\lambda v) - \frac{\partial q}{\partial \nu}(u) \right) \frac{\partial q}{\partial \nu}(u)\, d\sigma dt da \\
&+ 2\alpha\lambda \int_{\Sigma_0} uv\, d\sigma dt da
\end{aligned}$$

Or

$$\begin{aligned}
\frac{J(u+\lambda v) - J(u)}{\lambda} &= \lambda \left\| \frac{\frac{\partial q}{\partial \nu}(u+\lambda v) - \frac{\partial q}{\partial \nu}(u)}{\lambda} \right\|^2_{L^2(\Sigma_1)} + \alpha \lambda^2 \|v\|^2_{L^2(\Sigma_0)} \\
&+ 2\int_{\Sigma_1} \left(\frac{\frac{\partial q}{\partial \nu}(u+\lambda v) - \frac{\partial q}{\partial \nu}(u)}{\lambda} \right) \frac{\partial q}{\partial \nu}(u)\, d\sigma dt da \\
&+ 2\alpha \int_{\Sigma_0} uv\, d\sigma dt da
\end{aligned}$$

Passing to the limit when $\lambda \to 0$ in the previous equation, we obtain

$$\int_{\Sigma_1} \frac{\partial z}{\partial \nu} \frac{\partial q}{\partial \nu}\, d\sigma dt da + \alpha \int_{\Sigma_0} uv\, d\sigma dt da = 0. \tag{3.7}$$

Then $z = z(v) \in H^{2,1}(Q)$ is solution of (3.2). To interpret (3.7), we consider the adjoint state p solution of (2.2). If we multiply the first equation of (3.2) by p and integrate by parts over Q, we obtain:

$$\int_{\Sigma_1} \frac{\partial z}{\partial \nu} \frac{\partial q}{\partial \nu}\, d\sigma dt da = \int_{\Sigma_0} \frac{\partial p}{\partial \nu} v\, d\sigma dt da.$$

Combining the previous relation with (3.7), we have

$$\int_{\Sigma_0} (\frac{\partial p}{\partial \nu} + \alpha u) v\, d\sigma dt da = 0. \quad \forall v \in L^2(\Sigma_0).$$

Therefor

$$u = -\frac{\frac{\partial p}{\partial \nu}}{\alpha} \quad \text{on} \quad \Sigma_0.$$

Proposition 3.2. *Let $u \in L^2(\Sigma_0)$ be a solution of (1.6). Then we have the following estimate:*

$$\|u\|_{L^2(\Sigma_0)} \leq C\big(\|h_0\|_{L^2(\mathcal{O}^{TA})} + \|k\|_{L^2(\omega^{TA})} \big), \tag{3.8}$$

where $C = C(\alpha, T) > 0$.

Proof. In view of (3.7) and the decomposition of q given by (3.1), we have

$$
\begin{aligned}
0 &= \int_{\Sigma_1} \frac{\partial z(v)}{\partial \nu}\frac{\partial q}{\partial \nu}\, d\sigma dt da + \alpha \int_{\Sigma_0} uv\, d\sigma dt da,\\
0 &= \int_{\Sigma_1} \frac{\partial z(v)}{\partial \nu}\frac{\partial z(u)}{\partial \nu}\, d\sigma dt da + \int_{\Sigma_1} \frac{\partial z(v)}{\partial \nu}\frac{\partial \tau(k)}{\partial \nu} d\sigma dt da + \alpha \int_{\Sigma_0} uv\, d\sigma dt da,\\
0 &= a(v,u) + \int_{\Sigma_1} \frac{\partial z(u)}{\partial \nu}\frac{\partial \tau(k)}{\partial \nu} d\sigma dt da \quad \forall v \in L^2(\Sigma_0),
\end{aligned} \tag{3.9}
$$

where

$$
a(v,u) = \int_{\Sigma_1} \frac{\partial z(v)}{\partial \nu}\frac{\partial z(u)}{\partial \nu}\, d\sigma dt da + \alpha \int_{\Sigma_0} vw\, d\sigma dt da \quad \forall v,u \in L^2(\Sigma_0), \tag{3.10}
$$

is a bilinear form define on $L^2(\Sigma_0) \times L^2(\Sigma_0)$.

For any $v \in L^2(\Sigma_0)$, the bilinear fom $a(.,.)$ is coercive because

$$
a(u,u) = \left\| \frac{\partial z(u)}{\partial \nu} \right\|^2_{L^2(\Sigma_1)} + \alpha \|u\|^2_{L^2(\Sigma_0)} \geq \alpha \|u\|^2_{L^2(\Sigma_0)}.
$$

Taking $v = u$ in (3.9) and using the coercivity of $a(.,.)$, we deduce that

$$
\begin{aligned}
\alpha \|u\|^2_{L^2(\Sigma_0)} &\leq \left\| \frac{\partial z(u)}{\partial \nu} \right\|^2_{L^2(\Sigma_1)} \left\| \frac{\partial \tau(k)}{\partial \nu} \right\|^2_{L^2(\Sigma_1)},\\
&\leq C \|z(u)\|_{H^{2,1}(Q)} \|\tau(k)\|_{H^{2,1}(Q)},
\end{aligned}
$$

Which in view of (3.2) and (3.3), then there exists $C = C(\alpha, T) > 0$ such that

$$
\|u\|_{L^2(\Sigma_0)} \leq C\Big(\|h_0\|_{L^2(\mathcal{O}^{TA})} + \|k\|_{L^2(\omega^{TA})} \Big).
$$

□

3.2 Proof of null controlability problem

In oder to solve the null controllability problem, we need an inequality called an appropriate Carleman observability inequality associated with the adjoint of systems (2.1)-(2.3).

3.2.1 Carleman inequality

Carleman inequalities are employed to provide strong estimates that are vital for proving the controllability of certain partial differential equations. In particular, they are used to establish an observability estimate, which is essential for demonstrating that the system can be driven to a zero state. For this purpose, we define the following weight functions.

Lemma 3.1. *Let ω be an arbitrary non emphty open set of Ω. Then there exists $\psi \in C^2(\overline{\Omega})$ such that:*

$$
\begin{cases}
\psi(x) & > 0 \quad \forall x \in \Omega,\\
\psi(x) & = 0 \quad \forall x \in \Gamma,\\
\nabla\psi(x) & \neq 0 \quad \forall\, x \in \overline{\Omega - \omega_2}.
\end{cases} \tag{3.11}
$$

Let $\lambda \geq 0$ a real number. For any $(t, a, x) \in Q$, we set:

$$\varphi(t,a,x) = \frac{e^{2\lambda\|\psi\|_\infty} - e^{\lambda\psi(x)}}{t(T-t)a(A-a)}, \tag{3.12}$$

$$\eta(t,a,x) = \frac{e^{\lambda\psi(x)}}{t(T-t)a(A-a)}. \tag{3.13}$$

Remark 3.1. For $x \in \Gamma$, we have $\psi(x) = 0$; therefore, the function φ and η defined respectively by (3.12) and (3.13) depends on t and a, and we have

$$\varphi^{-1} \leq C(T, A). \tag{3.14}$$

Set

$$W(Q) = \left\{ \; z | z; \frac{\partial z}{\partial x_i} \in L^2(Q), i = 1......n; \frac{\partial z}{\partial t} \in L^2((0,T)\times(0,A); H^{-1}(\Omega)) \; \right\}.$$

We consider the following system

$$\left\{ \begin{array}{rcll} Lz &=& f & \text{in} \quad Q, \\ z &=& g & \text{on} \quad \Sigma, \end{array} \right. \tag{3.15}$$

with

$$g \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma).$$

We give now the global carleman inequality for system (3.15)

Proposition 3.3. *[16] Let also ψ, φ and η be respectively defined by* (3.11), (3.12) *and* (3.13). *Let ω be a non empty subset of Ω. Then there exists a positive constants $\lambda_0 \geq 1$ and $s_0 \geq 1$ and there exists a constant $C_1 > 0$ independant of $s \geq s_0$ and $\lambda \geq \lambda_0$ such that for any $s \geq s_0(\lambda)$ and $\lambda \geq \lambda_0$ and for $z \in W(Q)$ solution of* (3.15), *we have*

$$\mathcal{I}(z) \leq C_1\left(s^{-\frac{1}{2}} \|\varphi^{-\frac{1}{4}} g e^{-s\eta}\|^2_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} + s\lambda^2 \int_0^T \int_0^A \int_{\omega_2} e^{-2s\eta}\varphi \mid z \mid^2 dxdtda \right), \tag{3.16}$$

where

$$\mathcal{I}(z) = \int_Q \frac{1}{s\varphi} e^{-2s\eta} \mid \nabla z \mid^2 dxdtda + s\lambda^2 \int_Q \varphi e^{-2s\eta} \mid z \mid^2 \; dxdtda, \tag{3.17}$$

and

$$s_0(\lambda) = C(\psi)\frac{TA}{4} e^{2\lambda\|\psi\|_\infty}\left(\frac{T^2A^2}{4} + T^2A^3 + T^3A^2 + T + A\right).$$

For any $\rho^0 \in L^2(\Omega^A)$. We consider the following systems

$$\left\{ \begin{array}{rcll} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial \rho}{\partial a} - \Delta\rho + \mu\rho &=& 0 & \text{in} \quad Q, \\ \rho(t,a,x) &=& \dfrac{\partial \varpi}{\partial \nu}\chi_{\Gamma_1} & \text{on} \quad \Sigma, \\ \rho(0,a,x) &=& \rho^0 & \text{in} \quad \Omega^A, \\ \rho(t,0,x) &=& 0 & \text{in} \quad \Omega^T, \end{array} \right. \tag{3.18}$$

where ϖ solution of

$$\left\{ \begin{array}{rcll} -\dfrac{\partial \varpi}{\partial t} - \dfrac{\partial \varpi}{\partial a} - \Delta\varpi + \mu\varpi &=& 0 & \text{in} \quad Q, \\ \varpi(t,a,x) &=& -\dfrac{\frac{\partial \rho}{\partial \nu}}{\alpha}\chi_{\Gamma_0} & \text{on} \quad \Sigma, \\ \varpi(T,a,x) &=& 0 & \text{in} \quad \Omega^A, \\ \varpi(t,A,x) &=& 0 & \text{in} \quad \Omega^T, \end{array} \right. \tag{3.19}$$

Using Proposition 3.3 we have following results:

Proposition 3.4. *Under the assumption of the Proposition 3.3 are verified, there exists a constant C such that the following estimate is verified for all ρ solution of* (3.18) *and* (3.19)

$$\int_Q \frac{1}{\theta^2} \mid \rho \mid^2 \; dxdtda \leq C(C_1, \alpha, T)\|\rho\|^2_{L^2(\omega^{TA})}. \tag{3.20}$$

Proof. Applying the inequality (3.16) for ρ solution of (3.18)-(3.19), we have that

$$\mathcal{I}(\rho) \leq C_1\left(s^{-\frac{1}{2}} \left\|\varphi^{-\frac{1}{4}} e^{-s\eta}\frac{\partial \varpi}{\partial \nu}\right\|^2_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} + s\lambda^2 \int_0^T\int_0^A\int_\omega e^{-2s\eta}\varphi \mid \rho \mid^2 \; dxdtda\right). \tag{3.21}$$

In view of the definition of ψ, φ and η given respectively by (3.11), (3.12), (3.13) and the continuity of the trace, we have that there exists a positive constant $C(T)$ such that

$$\left\|(s\varphi)^{-\frac{1}{4}} e^{-s\eta}\frac{\partial \varpi}{\partial \nu}\right\|^2_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} \leq C(T)\|\varpi\|^2_{H^{2,1}(Q)}. \tag{3.22}$$

Using the oder hand that ϖ satisfies (3.19), (1.3) we deduce that

$$\|\varpi\|^2_{H^{2,1}(Q)} \leq C(\alpha, T)\left\|\frac{\partial \rho}{\partial \nu}\right\|^2_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma_0)}. \tag{3.23}$$

On the one hand the fact that

$$\left\|\frac{\partial \rho}{\partial \nu}\right\|^2_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma_0)} \leq C(T)\|\rho\|^2_{H^{2,1}(Q)}, \tag{3.24}$$

Combinig (3.22) - (3.24), using the fact that $\omega \subset \Omega$, and the Poincaré inequality we have that

$$\left\|(s\varphi)^{-\frac{1}{4}} e^{-s\eta}\frac{\partial \varpi}{\partial \nu}\right\|^2_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} \leq C(\alpha, T)\|\rho\|^2_{L^2(\omega^{TA})}. \tag{3.25}$$

In view of (3.21) and the fact that $s, \lambda \geq 1$ it follows from (3.25) that

$$\int_Q e^{-2s\eta}\varphi \mid \rho \mid^2 \; dxdtda \leq C(C_1, \alpha, T)\|\rho\|^2_{L^2(\omega^{TA})} + C_1\int_0^T\int_0^A\int_\omega e^{-2s\eta}\varphi \mid \rho \mid^2 \; dxdtda.$$

Since $e^{-2s\eta}\varphi \in L^\infty(Q)$, we deduce:

$$\int_Q e^{-2s\eta}\varphi \mid \rho \mid^2 \; dxdtda \leq C(C_1, \alpha, T)\|\rho\|^2_{L^2(\omega^{TA})}. \tag{3.26}$$

Set

$$\frac{1}{\theta^2} = e^{-2s\eta}\varphi. \tag{3.27}$$

Then according to the definition of η and φ, we have $\frac{1}{\theta^2} \in L^\infty(Q)$ and (3.26) can be rewritten

$$\int_Q \frac{1}{\theta^2} \mid \rho \mid^2 \; dxdtda \leq C(C_1, \alpha, T)\|\rho\|^2_{L^2(\omega^{TA})}.$$

□

Now, we are interested in the following null controllability problem: Find a control $k \in L^2(\omega^{TA})$ such that if $q = q(t,x;u(k),k) \in H^{2,1}(Q)$ is solution of (2.1)-(2.3), then (1.7) hods true.

If we multipliy the first line in (2.1) by ρ solution of (3.18) and the first line in (2.2) by Ψ solution of (3.19), and integrate by parts over Q, we obtain successively

$$\int_{\Omega^A} q(0,.,.)\rho^0 dxda - \int_{\Sigma_1} \frac{\partial q}{\partial \nu}\frac{\partial \varpi}{\partial \nu}\, d\sigma dtda - \frac{1}{\alpha}\int_{\Sigma_0} \frac{\partial p}{\partial \nu}\frac{\partial \rho}{\partial \nu}\, dxdtda = \int_0^T\int_0^A\int_{\mathcal{O}} h_0\rho\, dxdtda + \int_0^T\int_0^A\int_{\omega} k\rho\, dxdtda, \tag{3.28}$$

and

$$\int_{\Sigma_1} \frac{\partial q}{\partial \nu}\frac{\partial \varpi}{\partial \nu}\, d\sigma dtda = -\frac{1}{\alpha}\int_{\Sigma_0} \frac{\partial p}{\partial \nu}\frac{\partial \rho}{\partial \nu}\, d\sigma dtda. \tag{3.29}$$

Combining (3.28) and (3.29), we obtain

$$\int_{\Omega^A} q(0,.,.)\rho^0 dxda = \int_0^T\int_0^A\int_{\mathcal{O}} h_0\rho\, dxdtda + \int_0^T\int_0^A\int_{\omega} k\rho\, dxdtda,$$

The null controllability property is equivalent to find a control $k \in L^2(\omega^{TA})$ such that for any $\rho^0 \in L^2(\Omega^A)$, we have

$$\int_0^T\int_0^A\int_{\mathcal{O}} h_0\rho\, dxdtda + \int_0^T\int_0^A\int_{\omega} k\rho\, dxdtda = 0. \tag{3.30}$$

To find such a control, we consider for any $\epsilon > 0$ and for any $\rho^0 \in L^2(\Omega^A)$, the following functional:

$$J_\epsilon(\rho^0) = \frac{1}{2}\int_0^T\int_0^A\int_{\omega} |\, \rho\,|^2\ \ dxdtda + \int_0^T\int_0^A\int_{\mathcal{O}} h_0\rho\, dxdtda + \epsilon\|\rho^0\|_{L^2(\Omega^A)}. \tag{3.31}$$

whre ρ and ψ are respectively solutions of (3.18) and (3.19). We need to prove that the functional has a minimun in $L^2(\Omega^A)$ and then the controlability of (2.1)-(2.3)

Proposition 3.5. *Assume that* (1.2) *and that* ω *be a non empty subsets of* Ω. *Let* $\theta \in L^\infty(Q)$ *a positive weight function to be define by* (3.27) *and* $h_0 \in L^2(\mathcal{O}^{TA})$ *be such that* $\theta h_0 \in L^2(\mathcal{O}^{TA})$. *Then there exists a unique* $\rho^0_\epsilon \in L^2(\Omega^A)$ *such that:*

$$J_\epsilon(\rho^0_\epsilon) = \inf_{\rho^0 \in L^2(\Omega^A)} J_\epsilon(\rho^0). \tag{3.32}$$

Moreover, if $\rho^0_\epsilon \neq 0$, *we have the following optimality condition*

$$0 = \int_0^T\int_0^A\int_{\omega} \rho_\epsilon\rho\, dxdtda + \int_{\mathcal{O}^{TA}} h_0\rho\, dxdtda + \epsilon\frac{\displaystyle\int_{\Omega^A}\rho^0_\epsilon\rho^0 dxda}{\|\rho^0_\epsilon\|_{L^2(\Omega^A)}}, \quad \forall\rho^0 \in L^2(\Omega^A), \tag{3.33}$$

where ρ_ϵ *and* ψ_ϵ *are respective solutions to* (3.18) *and* (3.19) *corresponding to* $\rho^0 = \rho^0_\epsilon$. *Moreover there exists a positive constant* $C = C(C_1, \alpha, T)$ *independent of* $\epsilon > 0$ *such that* $k_\epsilon = \rho_\epsilon$ *in* ω_2^{TA} *satisfies*

$$\|k_\epsilon\|_{L^2(\omega_2^{TA})} \leq \sqrt{C}\|\theta h_0\|_{L^2(\mathcal{O}^{TA})}. \tag{3.34}$$

Proof. It is clear that J_ϵ is continuous and strictly convex and coercive on $L^2(\Omega^A)$, that threre exists a unique point $\rho_\epsilon^0 \in L^2(\Omega^A)$ where the fonctionnal J_ϵ reaches its minimun.

Now, assume that $\rho_\epsilon^0 \neq 0$. In order to prove (3.33). We write the Euler Lagrange conditions which characterise the minimum $\rho_\epsilon^0 \in L^2(\Omega^A)$:

$$\lim_{\lambda \to 0} \frac{J_\epsilon(\rho_\epsilon^0 + \lambda\rho^0) - J_\epsilon(\rho_\epsilon^0)}{\lambda} = 0, \quad \forall \rho^0 \text{ in } L^2(\Omega^A) \tag{3.35}$$

After somes calculations (3.35) yields (3.33).

Let ρ_ϵ^0 be the solution of (3.32) and ρ_ϵ be the solution of (3.18)-(3.19) associated to ρ_ϵ^0. Let also q_ϵ, p_ϵ be the solution associated $k = k_\epsilon = \rho_\epsilon$ of systems (2.1) and (2.2) respectively:

$$\left\{\begin{array}{rcll} -\dfrac{\partial q_\epsilon}{\partial t} - \dfrac{\partial q_\epsilon}{\partial a} - \Delta q_\epsilon + \mu q_\epsilon & = & h_0\chi_{\mathcal{O}} + k_\epsilon\chi_\omega & \text{in} \quad Q, \\ q_\epsilon(t,a,x) & = & u_\epsilon\chi_{\Gamma 0} & \text{on} \quad \Sigma, \\ q_\epsilon(T,a,x) & = & 0 & \text{in} \quad \Omega^A, \\ q_\epsilon(t,A,x) & = & 0 & \text{in} \quad \Omega^T, \end{array}\right. \tag{3.36}$$

and

$$\left\{\begin{array}{rcll} \dfrac{\partial p_\epsilon}{\partial t} + \dfrac{\partial p_\epsilon}{\partial a} - \Delta p_\epsilon + \mu p_\epsilon & = & 0 & \text{in} \quad Q, \\ p_\epsilon(t,a,x) & = & \dfrac{\partial q_\epsilon}{\partial \nu}\chi_{\Gamma 1} & \text{on} \quad \Sigma, \\ p_\epsilon(0,a,x) & = & 0 & \text{in} \quad \Omega^A, \\ p_\epsilon(t,0,x) & = & 0 & \text{in} \quad \Omega^T, \end{array}\right. \tag{3.37}$$

with

$$u_\epsilon = -\frac{\frac{\partial p_\epsilon}{\partial \nu}}{\alpha} \quad \text{on} \quad \Sigma_0, \tag{3.38}$$

and

$$k_\epsilon = \rho_\epsilon \quad \text{in} \quad \omega^{TA}. \tag{3.39}$$

Multiplying the first equation of (3.37) and (3.38) respectively by ρ and ϖ respectively solution of (3.18) and (3.19), and proceeding by integration by parts, we obtain successively

$$\int_{\Omega^A} q_\epsilon(0,.,.)\rho^0 dxdtda - \int_{\Sigma_1} \frac{\partial q_\epsilon}{\partial \nu}\frac{\partial \varpi_\epsilon}{\partial \nu}\, dxdtda - \frac{1}{\alpha}\int_{\Sigma_0} \frac{\partial p_\epsilon}{\partial \nu}\frac{\partial \rho}{\partial \nu}\, dxdtda = \tag{3.40}$$

$$\int_0^T \int_0^A \int_{\mathcal{O}} h_0\rho\, dxdtda + \int_0^T \int_0^A \int_\omega k\rho\, dxdtda, \tag{3.41}$$

and

$$\int_{\Sigma_1} \frac{\partial q_\epsilon}{\partial \nu}\frac{\partial \varpi_\epsilon}{\partial \nu}\, d\sigma dtda = -\frac{1}{\alpha}\int_{\Sigma_0} \frac{\partial p_\epsilon}{\partial \nu}\frac{\partial \rho}{\partial \nu}\, d\sigma dtda. \tag{3.42}$$

Combining (3.40) and (3.42) and we use (3.33), we obtain

$$\int_{\Omega^A} \big(q_\epsilon(0,.,.) + \epsilon\frac{\rho_\epsilon^0}{\|\rho_\epsilon^0\|_{L^2(\Omega^A)}}\big)\rho^0 dxda = 0.$$

Hence

$$q_\epsilon(0,.,.) = -\epsilon \frac{\rho_\epsilon^0}{\|\rho_\epsilon^0\|_{L^2(\Omega^A)}}.$$

Therefore

$$\|q_\epsilon(0,.,.)\|_{L^2(\Omega^A)} = \epsilon. \tag{3.43}$$

Now, if we take $\rho^0 = \rho_\epsilon^0$ in (3.51), we obtain that:

$$\begin{aligned}\|k_\epsilon\|^2_{L^2(\omega_2^{TA})} = \int_0^T \int_0^A \int_\omega \mid \rho_\epsilon \mid^2 \ dxdtda = -\int_{\mathcal{O}^{T\mathcal{A}}} h_0 \rho \ dxdtda - \epsilon \|\rho_\epsilon^0\|_{L^2(\Omega^A)} \\ \leq \|\theta h_0\|_{L^2(\mathcal{O}^{TA})} \leq \left\| \frac{1}{\theta} \rho_\epsilon \right\|_{L^2(Q)}.\end{aligned}$$

It then folows from (3.19) that

$$\|k_\epsilon\|_{L^2(\omega_2^{TA})} \leq \sqrt{C} \|\theta h_0\|_{L^2(\mathcal{O}^{TA})}.$$

where $C = C(C_1, \alpha, T) > 0$

□

3.2.2 Proof of theorem(2.2)

We proceed in three steps

Step 1. *We give some a priori estimates on u_ϵ, q_ϵ and p_ϵ.*

In view of (2.4) *and* (3.34)*, we have that*

$$\|u_\epsilon\|_{L^2(\Sigma_0)} \leq C\big(\|h_0\|_{L^2(\mathcal{O}^{TA})} + \|\theta h_0\|_{L^2(\mathcal{O}^{TA})}\big), \tag{3.44}$$

where $C(C_1, \alpha, T) > 0$. Since q_ϵ, p_ϵ satisfy
(3.36)-(3.37)*, using* (3.34), (3.44), (1.3) *and* (1.4) *, we prove that*

$$\|q_\epsilon\|_{H^{2,1}(Q)} \leq C\big(\|h_0\|_{L^2(\mathcal{O}^{TA})} + \|\theta h_0\|_{L^2(\mathcal{O}^{TA})}\big), \tag{3.45}$$

$$\left\| \frac{\partial q_\epsilon}{\partial \nu} \right\|_{L^2(\Sigma_1)} \leq C\big(\|h_0\|_{L^2(\mathcal{O}^{TA})} + \|\theta h_0\|_{L^2(\mathcal{O}^{TA})}\big), \tag{3.46}$$

$$\|p_\epsilon\|_{H^{2,1}(Q)} \leq C\big(\|h_0\|_{L^2(\mathcal{O}^{TA})} + \|\theta h_0\|_{L^2(\mathcal{O}^{TA})}\big). \tag{3.47}$$

$$\left\| \frac{\partial p_\epsilon}{\partial \nu} \right\|_{L^2(\Sigma_0)} \leq C\big(\|h_0\|_{L^2(\mathcal{O}^{TA})} + \|\theta h_0\|_{L^2(\mathcal{O}^{TA})}\big), \tag{3.48}$$

where $C(C_1, \alpha, T) > 0$.

Step 2. *We study the convergence when $\epsilon \to 0$ of sequences (k_ϵ), (u_ϵ), (q_ϵ), (p_ϵ), $\left(\frac{\partial q_\epsilon}{\partial \nu}\right)$ and $\left(\frac{\partial p_\epsilon}{\partial \nu}\right)$. In view of (3.34), (3.43) and (3.44)-(3.48) we can extract sub-sequences still denoted (k_ϵ), (u_ϵ), (q_ϵ), (p_ϵ) and $\left(\frac{\partial q_\epsilon}{\partial \nu}\right)$ such that when $\epsilon \to 0$, we have:*

$$k_\epsilon \rightharpoonup \hat{k} \text{ weakly in } L^2(\omega^{TA}), \tag{3.49}$$

$$u_\epsilon \rightharpoonup u \text{ weakly in } L^2(\Sigma_0), \tag{3.50}$$

$$q_\epsilon \rightharpoonup q \text{ weakly in } H^{2,1}(Q), \tag{3.51}$$

$$\frac{\partial q_\epsilon}{\partial \nu} \rightharpoonup \beta_1 \text{ weakly in } L^2(\Sigma_1), \tag{3.52}$$

$$p_\epsilon \rightharpoonup p \text{ weakly in } L^2(Q), \tag{3.53}$$

$$\frac{\partial p_\epsilon}{\partial \nu} \rightharpoonup \beta_2 \text{ weakly in } L^2(\Sigma_0), \tag{3.54}$$

$$q_\epsilon(0,.,.) \to 0 \text{ strongly in } L^2(\Omega^A). \tag{3.55}$$

Moreover using the weak lower semi-continuity of the norm, we deduce from (3.44) and (3.50) that

$$\|u\|_{L^2(\Sigma_0)} \leq C(C_1, \alpha, T)\big(\|h_0\|_{L^2(\mathcal{O}^{TA})} + \|\theta h_0\|_{L^2(\mathcal{O}^{TA})}\big). \tag{3.56}$$

Now, if we multiply the first equation in (3.36) and (3.37) respectively by $\varphi \in \mathcal{D}(Q)$ such that $\frac{\partial \varphi}{\partial \nu} = 0$ on Σ and $\xi \in \mathcal{D}(Q)$ integrate by parts over Q, then take the limit when $\epsilon \to 0$ while using (3.49), (3.51) and (3.53), we obtain that

$$\int_Q \big(-\frac{\partial \varphi}{\partial t} - \Delta\varphi + a_0\varphi\big) q \, dxdtda = \int_Q \big(h_0\chi_{\mathcal{O}} + \hat{k}\chi_\omega\big)\varphi \, dxdtda,$$

$$\int_Q \big(\frac{\partial \xi}{\partial t} - \Delta\xi + a_0\xi\big) p \, dxdtda = 0,$$

which after an integration by parts over Q, gives

$$\int_Q \big(\frac{\partial q}{\partial t} - \Delta q + a_0 q\big)\varphi \, dxdtda = \int_Q \big(h_0\chi_{\mathcal{O}} + \hat{k}\chi_\omega\big)\varphi \, dxdtda,$$

$$\int_Q \big(-\frac{\partial p}{\partial t} - \Delta p + a_0 p\big)\xi \, dxdtda = 0,$$

Hence, we deduce that

$$\frac{\partial q}{\partial t} - \Delta q + a_0 q = h_0\chi_{\mathcal{O}} + \hat{k}\chi_\omega \quad \text{in} \quad Q \tag{3.57}$$

$$-\frac{\partial p}{\partial t} - \Delta p + a_0 p = 0 \quad \text{in} \quad Q. \tag{3.58}$$

Since $q, p \in H^{2,1}(Q)$ and $\Delta q, \Delta p \in H^{-1}((0,T) \times (0,A);\ L^2(\Omega))$, we deduce that $q|_\Sigma, p|_\Sigma$ and $\frac{\partial q}{\partial \nu}|_\Sigma, \frac{\partial p}{\partial \nu}|_\Sigma$ exist and belong to $H^{\frac{3}{2},\frac{3}{4}}(\Sigma) \subset L^2(\Sigma)$ and $H^{\frac{1}{2},\frac{1}{4}}(\Sigma) \subset L^2(\Sigma)$ respectively. One the other and observing that $q, p \in L^2((0,T) \times (0,A); H^2(\Omega))$ and $\frac{\partial q}{\partial t}, \frac{\partial p}{\partial t} \in L^2((0,T) \times (0,A); H^{-2}(\Omega))$, we deduce that $q(0,.,.), p(0,.,.)$ and $q(T,.,.), p(T,.,.)$ exist in $H^{-1}(\Omega^A)$ and $q(.,0,.), p(.,0,.)$ and $q(.,A,.), p(.,A,.)$ exist in $H^{-1}(\Omega^T)$.

So, if we multiply the first equation in (3.37) by $\varphi \in \mathcal{C}^\infty(\bar{Q})$ such that $\varphi(.,0,.) = 0$ in Ω^T and $\varphi = 0$ respectively on $\Sigma \setminus \Sigma_1$, then integrate by parts over Q. And passing to the limit when ϵ tends towards zero, while using (3.49)-(3.52) and (3.55), we obtain that

$$-\int_{\Sigma_1} \beta_1 \xi \; dtdad\sigma + \int_{\Sigma_0} \frac{\partial \varphi}{\partial \nu} u \; d\sigma dtda \int_Q \big(\frac{\partial \varphi}{\partial t} - \Delta\varphi + a_0\varphi\big) q \; dxdtda = \int_Q \big(h_0\chi_{\mathcal{O}} + \hat{k}\chi_\omega\big)\varphi \; dxdtda. \tag{3.59}$$

Integrating by parts this later equality (3.59), and we use (3.57), we obtain

$$\int_{\Omega^A} \varphi(0,.,.)q(0,.,.)dxda + \int_{\Omega^A} \varphi(T,.,.)q(T,.,.)dxda + \int_{\Omega^T} \varphi(.,A,.)q(.,A,.)dxda - \tag{3.60}$$

$$\int_{\Sigma_1} \big(\frac{\partial q}{\partial \nu} - \beta_1\big)\varphi \; d\sigma dtda + \int_{\Sigma_0} \big(u - q\big)\frac{\partial \varphi}{\partial \nu} \; d\sigma dtda = 0. \tag{3.61}$$

Choosing successively in (3.60), $\varphi(T,.,.) = 0 \; in \; \Omega \; \varphi(.,A,.) = 0 \; in \; \Omega^T$, $\varphi = 0$ on Σ_1 and $\dfrac{\partial \varphi}{\partial \nu} = 0$ on Σ_0, we succesively get

$$q = u \quad \text{on} \quad \Sigma_0. \tag{3.62}$$

and

$$\frac{\partial q}{\partial \nu} = \beta_1 \quad \text{on} \quad \Sigma_1. \tag{3.63}$$

$$q(.A.,.) = 0 \quad \text{in} \quad \Omega^T. \tag{3.64}$$

$$q(T,.,.) = 0 \quad \text{in} \quad \Omega^A. \tag{3.65}$$

Finally, it follows from (3.60) that

$$q(0,.,.) = 0 \quad \text{in} \quad \Omega^A. \tag{3.66}$$

Combining (3.52) and (3.63), we obtain

$$\frac{\partial q_\epsilon}{\partial \nu} \rightharpoonup \frac{\partial q}{\partial \nu} = \beta_1 \quad \text{on} \quad \Sigma_1. \tag{3.67}$$

Now, if we multiply the first equation in (3.37) by $\xi \in \mathcal{C}^\infty(\bar{Q})$ such that $\xi(T,.,.) = 0$ in Ω^A, $\xi(.,A,.) = 0$ in Ω^T and $\xi = 0$ on $\Sigma \setminus \Sigma_0$, integrating by parts over Q. And passing to the limit when ϵ tends towards zero, while using (3.53), (3.54) and (3.67), we obtain

$$\int_{\Sigma_1} \frac{\partial \xi}{\partial \nu}\frac{\partial q}{\partial \nu} \; d\sigma dtda - \int_{\Sigma_0} \beta_2 \xi \; d\sigma dtda + \int_Q \big(\frac{\partial \xi}{\partial t} - \Delta\xi + a_0\xi\big) p \; d\sigma dtda = 0, \tag{3.68}$$

Integrating by parts this later equality (3.68), and we use (3.58), we obtain

$$\int_{\Omega^A} \xi(0,.,.)p(0,.,.)dxda + \int_{\Omega^T} \xi(.,0,.)p(.,0,.)dxdt + \int_{\Sigma_0} \big(\frac{\partial p}{\partial \nu} - \beta_2\big)\xi \; d\sigma dtda + \int_{\Sigma_1} \big(\frac{\partial q}{\partial \nu} - p\big)\frac{\partial \xi}{\partial \nu} \; d\sigma dtda = 0. \tag{3.69}$$

$\forall \xi \in \mathcal{C}^\infty(\bar{Q})$ with $\xi = 0 \; on \; \Sigma \setminus \Sigma_0$, $\xi(T,.,.) = 0 \; in \; \Omega^A \; and \; \xi(.,A,.) = 0 \; in \; \Omega^T$.

Choosing successively in (3.69), $\xi(0,.,.) = 0$ *in* Ω^A, $\xi(.,0,.) = 0$ *in* Ω^T, $\xi = 0$ on Σ_1 and $\dfrac{\partial \xi}{\partial \nu} = 0$ on Σ_0, we succesively get

$$p = \frac{\partial q}{\partial \nu} \quad \text{on} \quad \Sigma_1. \tag{3.70}$$

and

$$\frac{\partial p}{\partial \nu} = \beta_2 \quad \text{on} \quad \Sigma_0. \tag{3.71}$$

Finally, it follows from (3.69) that

$$p(0,.,.) = 0 \quad \text{in} \quad \Omega^A. \tag{3.72}$$

$$p(.,0,.) = 0 \quad \text{in} \quad \Omega^T. \tag{3.73}$$

Combining (3.54) and (3.71), we have

$$\frac{\partial p_\epsilon}{\partial \nu} \rightharpoonup \frac{\partial p}{\partial \nu} = \beta_2 \text{ weakly in } L^2(\Sigma_0). \tag{3.74}$$

In view of (3.57), (3.58), (3.62), (3.64), (3.65), (3.70), (3.72) and (3.73) q and p satisfies respectively

$$\left\{ \begin{array}{rcll} -\dfrac{\partial q}{\partial t} - \dfrac{\partial q}{\partial a} - \Delta q + \mu q &=& h_0\chi_{\mathcal{O}} + \hat{k}\chi_\omega & \text{in} \quad Q, \\ q(t,a,x) &=& u\chi_{\Gamma 0} & \text{on} \quad \Sigma, \\ q(T,a,x) &=& 0 & \text{in} \quad \Omega^A, \\ q(t,A,x) &=& 0 & \text{in} \quad \Omega^T, \end{array} \right. \tag{3.75}$$

$$\left\{ \begin{array}{rcll} \dfrac{\partial p}{\partial t} + \dfrac{\partial p}{\partial a} - \Delta p + \mu p &=& 0 & \text{in} \quad Q, \\ p(t,a,x) &=& \dfrac{\partial q}{\partial \nu}\chi_{\Gamma 1} & \text{on} \quad \Sigma, \\ p(0,a,x) &=& 0 & \text{in} \quad \Omega^A, \\ p(t,0,x) &=& 0 & \text{in} \quad \Omega^T, \end{array} \right. \tag{3.76}$$

From (3.38), (3.50) and (3.74), we obtain

$$u = -\frac{\frac{\partial p}{\partial \nu}}{\alpha} \quad \text{on} \quad \Sigma_0, \tag{3.77}$$

Finally, using the weak-lower semi-continuity of the norm and (3.49), we deduce from (3.34) the estimate (2.8)

Step 3. *Observing that* $k_\epsilon = \rho_\epsilon$ *in* ω^{TA}, *where* ρ_ϵ *satisfies*

$$\left\{ \begin{array}{rcll} \dfrac{\partial \rho_\epsilon}{\partial t} + \dfrac{\partial \rho_\epsilon}{\partial a} - \Delta \rho_\epsilon + \mu \rho_\epsilon &=& 0 & \text{in} \quad Q, \\ \rho_\epsilon(t,a,x) &=& \dfrac{\partial \varpi_\epsilon}{\partial \nu}\chi_{\Gamma 1} & \text{on} \quad \Sigma, \\ \rho_\epsilon(0,a,x) &=& \rho^0 & \text{in} \quad \Omega^A, \end{array} \right. \tag{3.78}$$

where ϖ_ϵ *solution of*

$$\left\{ \begin{array}{rcll} -\dfrac{\partial \varpi_\epsilon}{\partial t} - \dfrac{\partial \varpi_\epsilon}{\partial a} - \Delta \varpi_\epsilon + \mu \varpi_\epsilon &=& 0 & \text{in} \quad Q, \\ \varpi_\epsilon(t,a,x) &=& -\dfrac{\frac{\partial \rho_\epsilon}{\partial \nu}}{\alpha}\chi_{\Gamma 0} & \text{on} \quad \Sigma, \\ \varpi_\epsilon(T,a,x) &=& 0 & \text{in} \quad \Omega^A, \\ \varpi_\epsilon(t,A,x) &=& 0 & \text{in} \quad \Omega^T, \end{array} \right. \tag{3.79}$$

Now, if we apply the carleman inequality (3.20) *to* ρ_ϵ, *we have*

$$\int_Q \frac{1}{\theta^2} \mid \rho_\epsilon \mid^2 \ dxdtda \leq C(C_1, \alpha, T) \|\rho_\epsilon\|^2_{L^2(\omega^{TA})}. \tag{3.80}$$

Using (3.80), (3.39) *and* (3.34), *we deduce that*

$$\left\| \frac{1}{\theta} \rho_\epsilon \right\|^2_{L^2(Q)} \leq C(C_1, \alpha, T) \|\theta h_0\|^2_{L^2(\mathcal{O}^{TA})}. \tag{3.81}$$

Hence, if we set $L^2(\frac{1}{\theta}, Q) = \{z \in L^2(X), \int_X \frac{1}{\theta^2} \mid z \mid^2 dX < \infty\}$, *we deduce from this latter inequality that* ρ_ϵ *is bounded in* $L^2(\frac{1}{\theta}, Q)$. *Consequently, there exists* $\rho \in L^2(\frac{1}{\theta}, Q)$ *and a subsequence of* (ρ_ϵ) *still denoted* (ρ_ϵ) *such that*

$$\rho_\epsilon \rightharpoonup \hat{\rho} \quad \textit{weakly in} \quad L^2(\frac{1}{\theta}, Q). \tag{3.82}$$

If we refer to the definition of ψ, φ *and* η *given by* (3.11)-(3.13) *and the definition of* θ *given by* (3.27), *we can see that for all* $\tau > 0$,

$$\rho_\epsilon \rightharpoonup \hat{\rho} \quad \textit{weakly in} \quad L^2([\tau, T-\tau] \times [\tau, A-\tau] \times \Omega). \tag{3.83}$$

This implies that

$$\rho_\epsilon \rightharpoonup \hat{\rho} \quad \textit{weakly in} \quad \mathcal{D}'(Q). \tag{3.84}$$

Where $\mathcal{D}'(Q)$ *is the dual of* $\mathcal{D}(Q)$. *Therefore, it follows from* (3.34), (3.39) *and* (3.49)

$$\rho_\epsilon \rightharpoonup \hat{\rho} \quad \textit{weakly in} \quad L^2(\omega^{TA}), \tag{3.85}$$

In view of (3.39), (3.49) *and* (3.85), *we have*

$$\hat{k} = \hat{\rho} \textit{ in } \omega^{TA}. \tag{3.86}$$

It follows from (3.81) *we deduce that*

$$\left\| \frac{\partial \rho_\epsilon}{\partial \nu} \right\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma_0)} \leq C(C_1, \alpha, T) \|\theta h_0\|_{L^2(\mathcal{O}^{TA})}. \tag{3.87}$$

In view (3.79) *and* (1.3), *we obtain*

$$\|\varpi_\epsilon\|_{H^{2,1}(Q)} \leq C(C_1, \alpha, T) \left\| \frac{\partial \rho_\epsilon}{\partial \nu} \right\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma_0)}.$$

It follows from the continuite of the trace, we obtain respectively

$$\|\varpi_\epsilon\|_{H^{2,1}(Q)} \leq C(C_1, \alpha, T) \|\theta h_0\|_{L^2(\mathcal{O}^{TA})}, \tag{3.88}$$

and

$$\left\| \frac{\partial \varpi_\epsilon}{\partial \nu} \right\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma_0)} \leq C(C_1, \alpha, T) \|\theta h_0\|_{L^2(\mathcal{O}^{TA})} \tag{3.89}$$

Consequently, there exist $\hat{\varpi} \in H^{2,1}(Q)$ *and* $\beta_3 \in L^2(\Sigma_0)$ *such that*

$$\varpi_\epsilon \rightharpoonup \hat{\varpi} \quad \text{weakly in} \quad H^{2,1}(Q), \tag{3.90}$$

$$\frac{\partial \rho_\epsilon}{\partial \nu} \rightharpoonup \beta_3 \quad \text{weakly in} \quad L^2(\Sigma_0). \tag{3.91}$$

$$\frac{\partial \varpi_\epsilon}{\partial \nu} \rightharpoonup \beta_4 \quad \text{weakly in} \quad L^2(\Sigma_1). \tag{3.92}$$

Proceeding as for the convergence of q_ϵ and p_ϵ in Step 2, while passing to the limit in (3.78)*, we prove using* (3.91),(3.92) *and* (3.85)*, we prove that*

$$\frac{\partial \rho_\epsilon}{\partial \nu} \rightharpoonup \frac{\partial \hat{\rho}}{\partial \nu} = \beta_3 \text{ weakly in } L^2(\Sigma_0). \tag{3.93}$$

$$\hat{\rho} = \beta_4 \text{ on } \Sigma_1. \tag{3.94}$$

Now, passing to the limit in (3.79)*, we prove using* (3.90)- (3.92)*, $\hat{\varpi}$ is solution of* (2.7) *and*

$$\frac{\partial \hat{\varpi}}{\partial \nu} = \beta_4 \text{ on } \Sigma_1. \tag{3.95}$$

Combining (3.92) *and* (3.95)*, we have*

$$\frac{\partial \varpi_\epsilon}{\partial \nu} \rightharpoonup \frac{\partial \hat{\varpi}}{\partial \nu} = \beta_4 \text{ weakly in } L^2(\Sigma_1). \tag{3.96}$$

and

$$\hat{\rho} = \frac{\partial \hat{\varpi}}{\partial \nu} \text{ on } \Sigma_1. \tag{3.97}$$

We deduce that $\hat{\rho}$ is a solution of (2.6)*. If we take* (3.34) *and* (3.49)*, we deduce* (2.5)*. In view of* (3.66)*,* (3.75) *-* (3.77)*, we have that $(\hat{k}, q, p)$ is solution of the null controlability problems* (2.1)-(2.3)

4 Discussion

The paper discusses a Stackelberg control problem applied to a two-stroke linear system, particularly related to population dynamics with spatial and age structure. The system is managed by two hierarchical controls: the follower, responsible for boundary control aimed at minimizing the system's flow, and the leader, solving a null controllability problem to bring the system's state to zero at the initial time. The approach applies Carleman inequalities to prove controllability and optimal control in systems described by partial differential equations. The Stackelberg strategy is traditionally applied in economic contexts but adapted here to manage invasive species in structured environments. This demonstrates the strategy's flexibility beyond its traditional use. The implications are profound for systems requiring hierarchical decision-making, such as ecological management and biological systems, where spatial-temporal factors are vital. The novelty lies in using boundary and distributed controls in an integrated manner to handle non-homogeneous Dirichlet conditions.

5 Conclusion and Future Work

Using hierarchical control, the system achieves null controllability in the Stackelberg sense, with well-defined optimal controls. This paves the way for exploring more complex hierarchical control systems in future work. Specifically, it aims to address a Stackelberg-Nash control problem, which introduces more complex dynamics by involving multiple followers and a leader, each with distinct control objectives. This would further test the robustness and applicability of the Stackelberg strategy in controlling systems governed by partial differential equations. Future research aims to extend the Stackelberg approach to more complex systems, such as those involving multiple agents or missing data, while continuing to integrate economic and ecological control strategies.

Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of this manuscript.

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Competing Interests

Authors have declared that no competing interests exist.

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