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On Pairwise L-Closed Spaces

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

In this paper we define pairwise L-closed spaces and study their properties, we obtain several results concerning pairwise L-closed spaces, and some product theorems. Some examples dealing with pairwise L-closed spaces are discussed.

Keywords: Pairwise L-closed space; pairwise P-space; p-Lindelöf space; s-Lindelöf space; p-continuous function; p-homeomorphism function.

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1 Introduction

In mathematics, the notion of bitopological spaces is introduced and studied by J.C Kelly [1] in 1923, he defined pairwise Hausdorff, pairwise regular, pairwise normal spaces, and obtained generalizations of several standard results such as Urysohn's Lemma and Tietze's extension theorem.

Since then several mathematicians studied various concepts in bitopological spaces which turned to be an important field in general topology. We use $\mathbb{R}$ and $\mathbb{N}$ to denote the set of all real and natural numbers respectively, p- to denote pairwise and $\tau_{coc}, \tau_{dis}, \tau_s, \tau_u, \tau_r$ to denote cocountable, discrete, Sorgenfrey, usual and right ray topologies on $\mathbb{R}$ or $\mathbb{N}$. Also the τ_i-closure of a set A is denoted by $cl_i A$. Also we study the properties of pairwise L-closed spaces and their relations with other related concepts.

2 Preliminaries

Definition 2.1: A bitopological space (X, τ_1, τ_2) is said to be pairwise L-closed space if each τ_1-Lindelöf subset of X is τ_2-closed and each τ_2-Lindelöf subset of X is τ_1-closed.

Definition 2.2: A family $\widetilde{F}$ of non empty τ_1-closed subsets or τ_2-closed subsets of a bitopological space (X, τ_1, τ_2) is called a p-closed family if it contains at least one member F_1 and at least one member F_2 such that F_1 is τ_1-closed proper subset of X and F_2 is τ_2-closed proper subset of X. A family $\widetilde{F}$ of non empty subsets of X is $\tau_1\tau_2$-closed if every member of F is τ_1-closed or τ_2-closed (see 2.25 [2]).

Definition 2.3: A cover $\tilde{U}$ of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-open cover if $\tilde{U} \subseteq \tau_1 \cup \tau_2$, and it is called p-open cover for X if it contains at least one non empty member of τ_1 and at least one non empty member of τ_2 (see2.26 [2]).

Definition 2.4: A bitopological space (X, τ_1, τ_2) is said to be p-Lindelöf if every p-open cover for X has a countable subcover. Also X is called s-Lindelöf if every $\tau_1\tau_2$-open cover for X has a countable subcover (see 2.5 [2]).

Definition 2.5: A bitopological space (X, τ_1, τ_2) is τ_1-Lindelöf with respect to τ_2 if for each τ_1-open cover for X there is a countable τ_1-open subcover. Now if X is τ_1-Lindelöf with respect to τ_2 and it is τ_2-Lindelöf with respect to τ_1, then X is called B-Lindelöf (see 2.6 [2]).

Definition 2.6: A bitopological space (X, τ_1, τ_2) is called pairwise T_1 if for each pair of distinct points x, y in X, there exists a τ_1-neighbourhood U of x and a τ_2-neighbourhood V of y such that $x \in U$, $y \notin U$ and $y \in V, x \notin V$ (see 2.4 [3]).

Definition 2.7: A bitopological space (X, τ_1, τ_2) is called p-Hausdorff if $\forall x \neq y$ in X, there exists a τ_1-neighbourhood U of x and a τ_2-neighbourhood V of y such that $x \in U, y \in V$, and $U \cap V = \phi$(see 2.6 [3]).

Definition 2.8: [4] A bitopological space (X, τ_1, τ_2) is said to be a pairwise P-space if countable intersection of τ_1-open subsets of X is a τ_2-open subset of X and countable intersection of τ_2-open subsets of X is a τ_1-open subset of X. A point $x \in X$ is called a P-point if the intersection of countably many τ_1-neighborhoods of x is a τ_2-neighborhood of x, and the intersection of countably many τ_2-neighborhoods of x is a τ_1-neighborhood of x.

Definition 2.9: A bitopological space (X, τ_1, τ_2) is called second countable if (X, τ_1) is second countable and (X, τ_2) is second countable (see 2.7 [2]).

Definition 2.10: [2] A bitopological space (X, τ_1, τ_2) is called Lindelöf (resp.compact) if it is τ_1-Lindelöf (resp. τ_1-compact) and τ_2-Lindelöf (resp. τ_1-compact).

Example 2.11: Consider the bitopological space ($\mathbb{R}$,τ_u,τ_r) , let A=[0,1], then A is a τ_u-closed subset of $\mathbb{R}$. Furthermore A is τ_u-Lindelöf because ($\mathbb{R}$,τ_u) is Lindelöf. Now A is neither closed nor open in ($\mathbb{R}$,τ_r), hence ($\mathbb{R}$,τ_u, τ_r) is not a pairwise L-closed space.

Proposition 2.12: In a bitopological space (X, τ_1, τ_2), if every countable subset of X is closed, then every countable subset is discrete and every compact subset is finite (see 2.1 [3]).

Corollary 2.13: If (X, τ_1, τ_2) is a pairwise L-closed space, every countable subset of X is closed, discrete and every compact subset of X is finite.

Proposition 2.14: Every subspace of a pairwise L-closed space is pairwise L-closed.

Proof: Suppose that (X, τ_1, τ_2) is a pairwise L-closed space and Y is a subspace of it, let F be a τ_1-Lindelöf subset of Y, then F is a τ_1- Lindelöf subset of X, hence F is a τ_2-closed subset of X because X is a pairwise L-closed space. Similarly if we suppose that G is a τ_1-Lindelöf subset of Y, then G is τ_2-closed. Thus Y is a pairwise L-closed space.

Corollary 2.15 : If (X, τ_1, τ_2) is a p-Hausdorff pairwise P-space, then X is a pairwise L-closed space.

Proof: Let F be a τ_1-Lindelöf subset of X, let $x \in X$ such that $x \notin F$. Since (X, τ_1, τ_2) is p-Hausedorff, $\exists$ a sequence w_k of τ_1-open subsets such that $x \in \cap_{k=1}^{\infty} w_k$, also $\exists$ a sequence v_k of τ_2-open subsets such that $F \subseteq \cap_{k=1}^{\infty} v_k$, and $w_k \cap v_k = \phi$ $\forall k \in \mathbb{N}$. X is pairwise P-space, so $\cap_{k=1}^{\infty} w_k$ is τ_2-open subset containing x and $\cap_{k=1}^{\infty} w_k \cap F = \phi$,so F is a τ_2-closed subset of X. Similarly if we suppose that G is a τ_2-Lindelöf subset of X, we will get that it is τ_1-closed. Hence X is a pairwise L-closed space.

Proposition 2.16: Every Lindelöf pairwise L-closed bitopological space is a pairwise P-space.

Proof: Let (X, τ_1, τ_2) be a Lindelöf pairwise L-closed space, let $A = \cap_{k=1}^{\infty} u_k$ be a τ_1-G_δ set, then A is a τ_2-open subset of X since $X - A$=$X - \cap_{k=1}^{\infty} u_k$= $\cup_{k=1}^{\infty} (X - u_k)$ is a τ_1-F_σ-set, so $X - A$is a τ_1-Lindelöf subset of X because X is Lindelöf, but X is a pairwise L-closed space, so $X - A$ is a τ_2-closed subset of X. Hence A is a τ_2-open subset of X. Similarly, if we suppose that B is a τ_2-G_δ-set, we will get that it is a τ_1-open subset of X. Thus X is a pairwise P-space.

Corollary 2.17: For a p-Hausdorff Lindelöf bitopological space (X, τ_1, τ_2), X is a pairwise L-closed space if and only if it is pairwise P-space.

The proof follows from 2.15 and 2.16.

Definition 2.18: In a bitopological space (X, τ_1, τ_2), τ_1 is regular with respect to τ_2 if $\forall x \in X$ and each τ_1-closed set F such that $x \notin F$, there exists a τ_1-open set U and a τ_2-open set V such that $x \in U$ and $F \in V$ and $U \cap V$=ϕ.(see 4.3 [1])

Definition 2.19: [1] A bitopological space (X, τ_1, τ_2) is called p-regular if τ_1 is regular with respect to τ_2 and τ_2 is regular with respect to τ_1.

Definition 2.20: In a bitopological space (X, τ_1, τ_2), a point $x \in X$ has a pairwise L-closed neighborhood U if each τ_1-Lindelöf subset of U containing x is τ_2-closed, and each τ_2-Lindelöf subset of U containing x is τ_1-closed.

Proposition 2.21: Let (X, τ_1, τ_2) be a p-regular space. If every point in X has a pairwise L-closed neighborhood, then (X, τ_1, τ_2) is pairwise L-closed.

Proof: Let F be a τ_1-Lindelöf subset of X, let $x \in X$ such that $x \notin F$. If U is a τ_1-open subset containing x, then U is L-closed neighborhood. Since X is p-regular, $\exists$ a τ_1-open set H such that $x \in H \subseteq cl_2 H \subseteq U$ and $cl_2 H \cap F$ is a τ_1-Lindelöf subset of U, hence $cl_2 H \cap F$ is a τ_2-closed subset of U. $U - (cl_2 H \cap F)$ is a τ_2-open neighborhood of x, so $\{U - (cl_2 H \cap F)\} \cap F = \phi$ is a contradiction. Hence $x \in F$ and F is a τ_2-closed subset of X. Similarly if we assume that G is a τ_2-Lindelöf subset of X, by a similar argument we will get hat G is τ_1-closed. So $(X, \tau_1, \tau_2)$is a pairwise L-closed space.

Definition 2.22: A space (X, τ_1, τ_2) is said to be p-normal if for a τ_1-closed set C and a τ_1-closed set F such that $C \cap F = \phi$, there exist a τ_1-open set G, a τ_2-open set V such that $F \subseteq G$, $C \subseteq V$ and $V \cap G = \phi$.(see 2.18 [5])

Proposition 2.23: A p-regular pairwise L-closed space is p-normal.

Proof: Let (X, τ_1, τ_2) be a pairwise L-closed space, let A be a τ_1-Lindelöf subset of X and B be a τ_2-Lindelöf subset of X such that $A \cap B = \phi$, then A is τ_2-closed and B is τ_1-closed because X is pairwise L-closed. Since (X, τ_1, τ_2) is p-regular we have, $\forall a \in A$,$\exists$ a τ_1-closed subset F_a and a τ_2-open subset G_a such that $a \in G_a \subseteq F_a \subseteq X - B$. Now $\forall b \in B$,$\exists$ a τ_1-open subset C_b and τ_2-closed subset M_b such that $b \in C_b \subseteq M_b \subseteq X - A$. Let $\breve{U} = \{G_a : a \in A\}$ be a τ_2-open cover for A and $\widetilde{V} = \{C_b : b \in B\}$ be a τ_1-open cover for B. A and B are τ_1-Lindelöf and τ_2-Lindelöf respectively, so $A \subseteq \cup_{k=1}^{\infty} G_k$ and $B \subseteq \cup_{k=1}^{\infty} C_k$. Let $V_1 = C_1$ and for each positive integer $k > 1$, let $V_k = C_k - \cup_{j=1}^{k-1} F_j$. For each positive integer k, let $H_k = G_k - \cup_{j=1}^{k} Mj$ and $V = \cup_{k=1}^{\infty} V_k$, $H = \cup_{k=1}^{\infty} H_k$, then V is a τ_1-open subset of X and H is a τ_2-open subset of X. Also $A \subseteq V, B \subseteq H$. Furthermore, if $x \in H \cap V$, then $x \in H_i \cap V_l$ for some $i, l \in \mathbb{N}$, and so $x \in (G_i - \cup_{j=1}^{i} M_j) \cap (C_l - \cup_{j=1}^{l-1} F_j)$.

Consedering separately the cases $i > l$ and $i \leq l$ yields a contradiction and so $H \cap V = \phi$. Thus X is p-normal.

Definition 2.24 : A bitopological space (X, τ_1, τ_2) is said to be pairwise hereditarily Lindelöf if every τ_1-subspace of X is Lindelöf and τ_2-subspace of X is Lindelöf.

Corollary 2.25: For a pairwise hereditary Lindelöf bitopological space (X, τ_1, τ_2), the following are equivalent:

a. X is a pairwise L-closed space.
b. X is a countable discrete space.

Proposition 2.26: Every p-regular space which can be represented as a countable union of subspaces each of which has the pairwise L-closeness property has itself the pairwise L-closeness property.

Proof: Suppose that $X = \cup_{k=1}^{\infty} X_k$, X_k is pairwise L-closed subspace.Let A be a τ_i-Lindelöf subset of X_k for some $k \in \mathbb{N}$, then A is a τ_i-Lindelöf subset of X where X is p-regular. But A is a τ_j-closed subset of X_k because X_k is a pairwise L-closed subspace of X, hence A is a τ_j-closed subset of X $\forall i, j = 1, 2$ $i \neq j$. Thus X is pairwise L-closed.

Proposition 2.27: In the bitopological space (X, τ_1, τ_2), the sum $\oplus_{\alpha \in \Lambda} X_\alpha$ where X_α for some $\alpha \in \Lambda$ has a pairwise L-closeness property if and only if all spaces X_α have a pairwise L-closeness property.

Proof: $\Longrightarrow$) Suppose that the sum $\oplus_{\alpha \in \Lambda} X_\alpha$ where $X_\alpha \neq \phi$ for some $\alpha \in \Lambda$ is pairwise L-closed space, then X_α is a pairwise L-closed subset of $\oplus_{\alpha \in \Lambda} X_\alpha$ because X_α is a closed subspaces of $\oplus_{\alpha \in \Lambda} X_\alpha$.

$\Longleftarrow$) Suppose that F is a τ_i-Lindelöf subset of $\oplus_{\alpha\in\Lambda}X_\alpha$, then $F\cap X_\alpha$ is a τ_i-Lindelöf subset of X_α. But X_α is pairwise L-closed, hence F is a τ_j-closed subset of $X_\alpha \forall\alpha\in\Lambda$,so F is a τ_j-closed subset of $\oplus_{\alpha\in\Lambda}X_\alpha$ $\forall\alpha\in\Lambda, i,j=1,2$ $i\neq j$. Thus $\oplus_{\alpha\in\Lambda}X_\alpha$ is pairwise L-closed.

Definition 2.28: [4] A bitopological space (X,τ_1,τ_2) is pairwise almost Lindelöf if every τ_i-open cover $\breve{U}$=$\{u_\alpha$: $\alpha\in\Lambda\}$of X has a countable subcollection $\breve{U}\prime$=$\{u_\alpha$:$\alpha\in\Lambda_1\subseteq\Lambda\}$ of Λsuch that X=$\cup_{\alpha\in\Lambda}cl_j\ u_\alpha$ $\forall i,j=1,2$ $i\neq j$.

Definition 2.29: A bitopological space (X,τ_1,τ_2) is called pairwise hereditarily almost Lindelöf if every subspace of X is pairwise almost Lindelöf.

Proposition 2.30: If (X,τ_1,τ_2) is a pairwise L-closed space, then the following are equivalent:

a. X is pairwise hereditarily almost Lindelöf.
b. X is pairwise hereditarily Lindelöf .
c. X is countable discrete.

Proof: c$\rightarrow$a) Suppose that X is a countable discrete space such that $X=\cup_{k\in\mathbb{N}}F_k$, F_k is a τ_i-Lindelöf subset of X. If $\breve{U}$=$\{u_\alpha$:$\alpha\in\Lambda\}$ is a τ_i-open cover for F_k and $\breve{U}\prime$=$\{u_\alpha$:$\alpha\in\Lambda_1\subseteq\Lambda\}$is a countable subcollection of $\Lambda$where u_α is a τ_i-open subset of X, then $F_k\subseteq\cup_{\alpha\in\Lambda1}u_\alpha$. But X is pairwise L-closed,hence F_k is τ_j-closed, and $X=\cup_{k\in\mathbb{N}}F_k\subseteq\cup_{\alpha\in\Lambda1}u_\alpha$ $\forall i,j=1,2i\neq j$.Thus X is pairwise hereditarily almost Lindelöf.

Proposition 2.31: Every $\tau_1\tau_2$-open cover for a p-regular pairwise L-closed space (X,τ_1,τ_2) has locally countable open refinement.

Proof: Suppose that $\breve{U}=\{u_x : x\in X\}$ is a $\tau_1\tau_2$-open cover for X. Since X is p-regular, $\forall x\in X$, there exists a τ_1-open set u_x such that $x\in v_x\subseteq cl_2v_x\subseteq u_x$ for some τ_1-open set v_x. Let $\{u_{x_k}$: $k\in\mathbb{N}\}$ be a countable subcover for $\breve{U}$. The sets $H_k=u_{x_k}-(cl_2v_{x_1}\cup cl_2v_{x_2}\cup\ldots)$ are τ_1-open or τ_2-open that constitude a $\tau_1\tau_2$-open cover for X. $\forall x\in X$ we have $x\in H_{k_{(x)}}$ where $^{k(x)}$ is the smallest integer such that $x\in u_{x_k}$. $\{H_k : k\in\mathbb{N}\}$ refines $\breve{U}$ and it is locally countable because $v_{x_k}\cap H_j=\phi$ $\forall j>k$.

3 Product Properties of Pairwise L-closed Spaces

Definition 3.1: If (X,τ_1,τ_2) and (Y,σ_1,σ_2) are two bitopological spaces, a function f:$(X,\tau_1,\tau_2)\rightarrow(Y,\sigma_1,\sigma_2)$ is said to be p-continuous if f_1:$(X,\tau_1)\rightarrow(Y,\sigma_1)$ is continuous and f_2:$(X,\tau_2)\rightarrow(Y,\sigma_2)$ is continuous.

Proposition 3.2 : Let (X,τ_1,τ_2) and (Y,σ_1,σ_2) be two bitoplogical spaces such that (Y,σ_1,σ_2) is a pairwise L-closed space, if f:$(X,\tau_1,\tau_2)\rightarrow(Y,\sigma_1,\sigma_2)$ is a p-continuous one to one function, then (X,τ_1,τ_2) is a pairwise L-closed space.

Proof: Suppose that f:$(X,\tau_1,\tau_2)\rightarrow(Y,\sigma_1,\sigma_2)$ is a p-continuous one to one function. Let (Y,σ_1,σ_2) be a pairwise L-closed space, let F be a τ_1-Lindelöf subset of X, then f(F) is σ_2-Lindelöf because f is a p-continuous function, but (Y,σ_1,σ_2) is a pairwise L-closed space, so f(F) is a σ_2-closed subset of Y. Now F=f^{-1}(f(F)) is a τ_2-closed subset of X since f is one to one. Similarly if we suppose that G is a τ_2-Lindelöf subset of X, we will get that it is τ_2-closed. Hence (X,τ_1,τ_2) is pairwise L-closed.

Definition 3.3: Let (X,τ_1,τ_2) and (Y,σ_1,σ_2) be two bitopological spaces, a function f: $(X,\tau_1,\tau_2)\rightarrow(Y,\sigma_1,\sigma_2)$ is called p-homeomorphism if f is bijection, p-continuous and f^{-1} is p-continuous. (X,τ_1,τ_2)

and (Y, σ_1, σ_2) are called p-homeomorphic.

Definition 3.4: A function f:$(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called pairwise open if the induced functions f_1:$(X, \tau_1) \to (Y, \sigma_1)$ and f_2:$(X, \tau_2) \to (Y, \sigma_2)$ are both open. A function f:$(X, \tau) \to (Y, \sigma)$ is closed if it sends closed sets onto closed sets. The function f:$(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$is called pairwise closed if the induced functions f_1:$(X, \tau_1) \to (Y, \sigma_1)$ and f_2:(X,τ_2)$\to$(Y,σ_2) are both closed (see 2.45 [2]).

Proposition 3.5: Let (X,τ_1,τ_2) be a p-Lindelöf bitopological space, (Y,σ_1,σ_2) is a pairwise L-closed space, if f: $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a bijection p-continuous function, then f is p-homeomorphism.

Proof: It suffices to show that f is a pairwise closed function. Let C be a τ_1-closed proper subset of X, since X is p-Lindelöf, C is a τ_2-Lindelöf subset of X (see 2.29 [2]), hence f(C) is σ_2-Lindelöf because f is p-continuous. But (Y, σ_1, σ_2) is a pairwise L-closed space,so f(C) is σ_1-closed. Similarly if we suppose that F is a τ_2-closed proper subset of X, we will get that f(F) is a σ_2-closed subset of Y. Hence f is a pairwise closed function and f is p-homeomorphism.

Corollary 3.6: If a p-continuous function from a p-Hausdorff Lindelöf bitopological space to a pairwise L-closed space is pairwise closed, then every p-continuous bijective function is p-homeomorphism.

Proposition 3.7: Pairwise L-closeness property is a bitopological property.

Proof: Let (X, τ_1, τ_2) be a pairwise L-closed space and (Y, σ_1, σ_2) be any bitopological space. Suppose that h: $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is p-homeomorphism, let A be a τ_1-Lindelöf subset of X, then h(A) is σ_1-Lindelöf because h is p-continuous. Since X is a pairwise L-closed space, A is τ_2-closed, hence h(A) is σ_2-closed because h is pairwise L-closed. Similarly if we suppose that B is τ_2-Lindelöf, we will get that h(B) is σ_1-closed. Thus Y is a pairwise L-closed space.

Remark 3.8: The product of two Lindelöf topological spaces need not to be Lindelöf. In general the product of two Lindelöf bitopological spaces is not necessarly Lindelöf as the following example shows (see 2.21 [2]).

Let $X = \mathbb{R} \times I$ where I is an interval, let "¡" be the lexicographical order in X. Let β_1=$\{[x, y)$: $x < y$ $x, y \in \mathbb{R}\}$be a base for the lower limit topology (or Sorgenfrey topology) $\tau 1$ on X and β_2=$\{(x, y]$: $x < y$ $x, y \in \mathbb{R}\}$ be a base for τ_2 on X, so (X, τ_1, τ_2) is a Lindelöf bitopological space. $(X \times X, \tau_1 \times \tau_1, \tau_2 \times \tau_2)$ is not $(\tau_1 \times \tau_1)$-Lindelöf because $(\tau_1 \times \tau_1)$-closed subspace L=$\{(x, y)$:$x = -y$ $x, y \in \mathbb{R}\}$ is not a $(\tau_1 \times \tau_1)$-Lindelöf subspace, it is discrete.

Proposition 3.9: If (X, τ_1, τ_2) and (Y, σ_1, σ_2) are pairwise L-closed bitopological spaces such that either X or Y is p-regular, then $X \times Y$ is a pairwise L-closed space.

Proof: Suppose that (X, τ_1, τ_2) and (Y, σ_1, σ_2) are pairwise L-closed spaces, let Y be p-regular, let F be a $(\tau_1 \times \sigma_1)$-Lindelöf subset of $X \times Y$. If $(x_\circ, y_\circ) \notin F$, so $(x_\circ, y_\circ) \notin [(\{x_\circ\} \times Y\} \cap F$ and $(\{x_\circ\} \times Y) \cap F$is a τ_2-closed subset of $X \times Y$ because Y is pairwise L-closed. Since Y is p-regular,$\exists$ a σ_1-open set H containing $y_\circ$ such that $(X \times cl_2 H) \subseteq \{X - (\{x_\circ\} \times Y) \cap F]$, so the projection function $\pi_x((X \times cl_2 H) \cap ((\{x_\circ\} \times Y) \cap F)$ is a τ_2-closed subset of X because π_x is p-continuous. $X - [\pi_x(X \times cl_2 H) \cap F) \times Y \cap (X \times H)]$is τ_2-open neighborhood of $(x_\circ, y_\circ)$ disjoint from F, hence F is $(\tau_2 \times \sigma_2)$-closed subset of $X \times Y$. Similarly if we suppose that G is a $(\tau_2 \times \sigma_2)$-Lindelöf subset of $X \times Y$, then it is a $(\tau_1 \times \sigma_1)$-closed subset of $X \times Y$. Therefore $X \times Y$ is pairwise L-closed.

Proposition 3.10: The product of two finite number of pairwise L-closed p-regular spaces is pairwise L-closed.

Proof: Let $\{X_k : k \in \mathbb{N}\}$ be a family of finitely many p-regular pairwise L-closed spaces. Let $X = X_k$, by induction on k, for $k = 2$ the result is given by 3.9. Suppose that the result is true for $k = n$ $\forall n \in \mathbb{N}$, we want to show that it is true for $k = n + 1$. Now $(X_1 \times X_2 \times \ldots \times X_n) \times X_{n+1}$ is p-homeomorphic to $X_1 \times X_2 \times \ldots \times X_n \times X_{n+1}$, so by induction hypothesis we get that $X_1 \times X_2 \times \ldots \times X_n \times X_{n+1}$ is pairwise L-closed. Hence X is a pairwise L-closed.

Definition 3.11: A surjective function f:$(X, \tau) \rightarrow (Y, \sigma)$ is a Lindelöf function if whenever K is a Lindelöf closed subset of Y, we have f$^{-1}(K)$ is a Lindelöf subset of X. A surjective function f:$(X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is called pairwise Lindelöf function if the induced function f:$(X, \tau_i) \rightarrow (Y, \sigma_i)$ is Lindelöf function $\forall i = 1, 2$.

Proposition 3.12: Let (X, τ_1, τ_2) be a pairwise L-closed space, and (Y, σ_1, σ_2) be a Lindelöf space, then $\pi_x : X \times Y \rightarrow X$ is a pairwise Lindelöf function.

Proof: Let F be a τ_1-Lindelöf subset of X, then F is a τ_2-closed subset of X because X is pairwise L-closed. The projection function $\pi_{x|F\times Y}$ is pairwise-closed such that $(\pi_{x|F\times Y}\ )^{-1}(x)$ is τ_1-Lindelöf because π_x is p-continuous. Similarly if we suppose that G is τ_2-Lindelöf, we will get that $(\pi_{x|F\times Y}\ )^{-1}(x)$ is τ_2-Lindelöf. Hence Lindelöf is a pairwise Lindelöf function.

Proposition 3.13: Let (X, τ_1, τ_2) be a pairwise L-closed space and (Y, σ_1, σ_2) be any bitopological space. If f: $(X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is any pairwise function and $\{(x,\text{f}(x)) : x \in X\}$ is p-Lindelöf, then f is p-continuous.

Proof: Let π_x and π_y be two projection functions, then X and f(X) are two Lindelöf sets as images of Lindelöf sets under π_x and π_y. Let $\pi_{x\prime} = \pi_{x|f}$, then $\pi_{x\prime}$ is a pairwise closed projection function...(1) and this is because if $A \subseteq f(X)$ is τ_i-closed subset, then A is τ_j-Lindelöf where f(X) is Lindelöf ∀i,j=1,2 i≠j. So $\pi_{x\prime}(A)$ is p-Lindelöf p-closed because X is a pairwise L-closed space. Since f is defined on X, $\pi_{x\prime}$ is a bijection function. . . (2). From (1) and (2) we get $\forall \tau_i$-open set $v \subseteq$f we have $\pi_{x\prime}(v)$ is τ_i-open in X. Hence f $=\pi_y \circ (\pi_{x\prime})^{-1}$ is p-continuous.

Proposition 3.14: If (X, τ_1, τ_2) and (Y, σ_1, σ_2) are p-Hausdorff pairwise L-closed spaces, then $(X \times Y, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2)$ is a $\tau_i \times \sigma_i$-L-closed space ∀i=1,2.

Definition 3.15: Let (X, τ) be a topological space and $A \subset X$. If for every neighborhood U_x of $x \in X$ we have $|U_x \cap A| = |A|$, then x is called a complete accumulation point of A.

Proposition 3.16: If $(X, \tau_1, \tau_2)$is a pairwise L-closed space and A is a τ_i-Lindelöf subset of X such that $|A| = \omega_1$ ∀i=1,2, if x is an accumulation point of A, then x is a complete accumulation point.

Proof: Let A be a τ_i-Lindelöf subset of X such that $|A| = \omega_1$, then A is τ_j-Lindelöf-closed ∀i,j=1,2 i≠j because X is pairwise L-closed. Let x be an accumulation point of A, hence $x \in cl_{\text{j}}A = A$. Let O_x be a τ_j-Lindelöf-neighborhood of x, if we take f:$A \cap O_x \rightarrow A$ defined by f(x)=x , then f is a p-continuous one to one function. Hence $|O_x \cap A| = |A|$.

Competing Interests

Authors have declared that no competing interests exist.

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