Global Analysis of a Directed Dynamics Competition Model

Md. Kamrujjaman[1*] and Kamrun Nahar Keya[1]

[1]*Department of Mathematics, University of Dhaka, Dhaka 1000, Bangladesh.*

Authors' contributions

This work was carried out in collaboration between both authors. Author MK designed the study, carried out the model analysis, literature searches and wrote the initial and final draft of the manuscript. Author KNK managed numerical simulation of the study, partial analysis of the model and gave insightful comments on all sections. Both authors read and approved the final manuscript.

Original Research Article

Abstract

In this study, we consider a directed-diffusion system describing the interactions between two organisms in heterogeneous environment. We focus on the effects of two distribution functions while two species are distributed with their corresponding resource function. We determine the global asymptotic stability of semi-trivial as well as the coexistence steady states due to interactions among three smooth functions.

*Corresponding author: E-mail: kamrujjaman@du.ac.bd;

1   Introduction

The study of population dynamics in mathematics started with the description of total population size and its dynamical behaviour. Historically, some of the applications connected with mathematics to biology were in the area of population dynamics; the instantaneous examples are

1. the unbounded exponential growth predicted by Malthus,

2. the logistic law proposed by Verhulst which corresponds to the limited resources of the environment, and

3. the two more growth functions such as Gilpin-Ayala [1], extended form of logistic and Gompertz law [2].

In many cases, either spatially distributed or dependence on the upward history of population densities cannot be neglected, thus the adequate mathematical models would be either partial differential or functional differential equations or sometimes both. In this paper, we consider a system of partial differential equations that present the directional dynamics where species movement are depending on their own resource functions, respectively, see similar model in [3].

At least in the last two decades, the competition model with regular diffusion was considered in the literature [4, 5, 6, 7, 8, 9] and references therein. For spatial heterogeneity of the environment in ecology, the space movements are modeled with diffusions, and a certain effort was undertaken to explain the role of dispersal coefficients in non-homogeneous environment. Introducing spatial distribution of species in mathematical description of population dynamics aims to explain certain real world phenomena. Dockery *et al.* in [6], presented an illustrative example of the fact that combined effects of dispersion and spatial non-homogeneity is that the slowest diffuser always the sole winners. In this study, they consider the number of n phenotypes competing for the natural resources.

Interaction of two species, either competing or cooperating, is usually described by the Lotka-Volterra model. Generally, the inclusion of a standard diffusion term (notationally, $d\Delta u$) leads to the uniform ideal free distribution while the diffusion coefficient is too high, i.e. $d \to \infty$, which is not feasible for problems when the carrying capacity is space-dependent. If we have a Lotka system with partially shared resources, the situation changes. There is an asymptotically stable steady state once the difference between diffusion rates is not very significant. In [9], the interesting observation is the evolutionary advantage of spatial carrying capacity compared to the non-spatial carrying capacity with the same average value over space. For diffusing populations, not only the diffusion speed rate but also the strategy has become an object of intensive discussion [10, 3, 11].

The present paper is addressed in the following way: considering two species with similar diffusion strategy mostly concerned with the case when they compete for similar basic resources, and their diffusion strategies follow the dispersal towards two prescribed positive distribution, and they are individual for each of the two organisms. We now consider the competition model with directed dynamics and homogeneous Neumann boundary conditions:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta \left(\dfrac{u(t,x)}{P(x)} \right) + u(t,x)(K(x) - u(t,x) - v(t,x)), \ t > 0, \ x \in \Omega, \\[2mm] \dfrac{\partial v}{\partial t} = d_2 \Delta \left(\dfrac{v(t,x)}{Q(x)} \right) + v(t,x)(K(x) - u(t,x) - v(t,x)), \ t > 0, \ x \in \Omega, \\[2mm] \dfrac{\partial (u/P)}{\partial n} = \dfrac{\partial (v/Q)}{\partial n} = 0, \ x \in \partial\Omega, \\[2mm] u(0,x) = u_0(x), \ v(0,x) = v_0(x), \ x \in \Omega. \end{cases} \tag{1.1}$$

Here $u(t,x)$, $v(t,x)$ represent the population density of two competing species which are therefore assumed to be non-negative, with corresponding migration rates d_1, d_2 respectively. The function $K(x)$ represents their common resource function or carrying capacity and two distributions are $P(x)$

and $Q(x)$ and all these functions are in the class of $C^{1+\alpha}$, $\alpha > 0$. Also the notation Ω is a bounded region in $\mathbb{R}^n$ while the smooth boundary is $\partial\Omega$, and n denotes the unit normal vector on $\partial\Omega$. The paper is organized as follows. In section 2, we present existence, and positivity of solutions for single species and established some results which will be used in later sections. The key ingredients of section 3 are to state different equilibrium states of the system (1.1) and justified and developed the main results. In section 4, we present some numerical examples exposing the relationship between average populations with various distributions; means the system has coexistence solution independently of diffusion speed and extinction of one by other as well. Finally, section 5 presents the conclusion and some discussions. Before starting the theoretical study of the system (1.1), we consider the following two arbitrary examples for graphical observation to find the connection between mathematics and biology.

Example 1. *Let us consider $P(x) \equiv Q(x) \equiv 1.3 + cos(\pi x)$ on $\Omega = (0, 1)$ with $d_1 = d_2 = 1$ and the initial density $(u_0, v_0) = (0.8, 1.8)$. If $K(x) \equiv P(x) + c$, both populations of (1.1) are coexisting as shown in Fig. 1 for particular values of $c > 0$. The figure shows the relation of the density of two species u and v for non-rational functions $P(x) \equiv Q(x)$ and $K(x)$.*
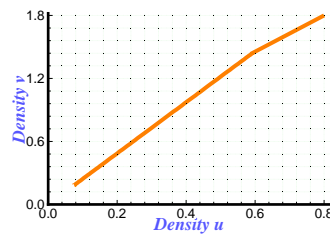
Fig. 1. Density of species $u(t, x)$ versus $v(t, x)$ at time $t = 100$ where
$P(x) \equiv Q(x) \equiv 1.3 + cos(\pi x)$**, $K(x) \equiv P(x) - c$, $c = 0.2$ and $u_0 = 0.8$, $v_0 = 1.8$**

Example 2. *If we vary initial densities sufficiently, from low to high with the same resource function as considered in Example 1 then the relation between two densities $u(t, x)$ and $v(t, x)$ are approximated and shown in the following Fig. 2. In both diagrams (a) and (b) of Fig. 2, visually it is observed that the species u and v are cooperating with each other but their initial density is important.*

Well, keeping these two examples under consideration, now we want to explore the characteristic behaviours of (1.1) in absence of one population.
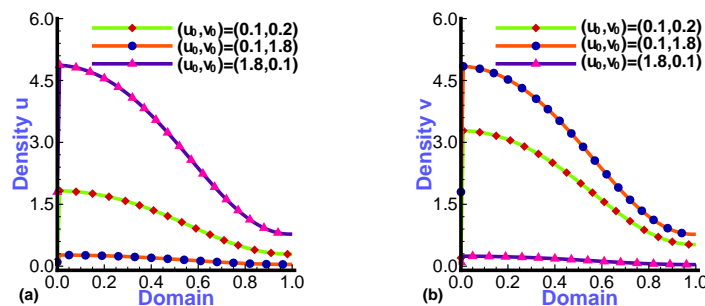
Fig. 2. Solutions of (1.1) for $P(x) \equiv Q(x) \equiv 1.3 + cos(\pi x)$, and $K(x) \equiv P(x) - c$, $c > 0$, $\Omega = (0, 1)$,$d_1 = d_2 = 1.0$ and for various initial conditions $(u_0, v_0) = (0.1, 0.2), (0.1, 1.8)$ and $(1.8, 0.1)$ (a) the variation of density $u(t, x)$, and (b) the variation of density $v(t, x)$ over space.

2 Analysis of Single Species Model

The function $u^*(x)$ is the stationary solution of the following boundary value problem and it corresponds to a situation where only species u survives in (1.1):

$$d_1 \Delta \left(\frac{u^*(x)}{P(x)} \right) + u^*(x)(K(x) - u^*(x)) = 0, \; x \in \Omega, \; \frac{\partial(u^*/P)}{\partial n} = 0, \; x \in \partial\Omega \qquad (2.1)$$

Proposition 1. *Suppose that $P(x) \not\equiv const$, $K(x) \not\equiv const$, and $P(x)$, $K(x)$ are linearly independent and let $u^*(x)$ be a positive solution of (2.1) then*

$$\int_\Omega P(x)(u^*(x) - K(x)) \, dx = d_1 \int_\Omega \frac{|\nabla(u^*/P)|^2}{(u^*/P)^2} \, dx \; > 0. \qquad (2.2)$$

Proof. Since $u^* > 0$ and $P(x) > 0$, dividing the first equation of (2.1) by u^*/P, we obtain

$$d_1 \frac{\Delta(u^*/P)}{(u^*/P)} + P(x)(K(x) - u^*(x)) = 0, \; x \in \Omega, \; \frac{\partial(u^*/P)}{\partial n} = 0, \; x \in \partial\Omega \qquad (2.3)$$

Integrating (2.3) over the domain Ω using boundary conditions in (2.3), we have

$$d_1 \cdot 0 + d_1 \int_\Omega \frac{|\nabla(u^*/P)|^2}{(u^*/P)^2} \, dx + \int_\Omega P(x)(K(x) - u^*(x)) \, dx = 0$$

which yields

$$\int_\Omega P(x)(u^*(x) - K(x)) \, dx = d_1 \int_\Omega \frac{|\nabla(u^*/P)|^2}{(u^*/P)^2} \, dx \; > 0 \text{ unless } u^*(x) = P(x) \qquad (2.4)$$

It is seen that $u^*(x) = P(x)$ is not a solution of (2.1) as far as $P(x)$ and $K(x)$ are non-proportional. $\qquad \square$

Proposition 2. *Suppose that $P(x) \not\equiv const$, $P(x)/K(x) \not\equiv const$ and let $u(t,x)$ be a positive solution to*

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = d_1 \Delta \left(\dfrac{u(t,x)}{P(x)} \right) + u(t,x)(K(x) - u(t,x)), \; t > 0, \; x \in \Omega, \\ \dfrac{\partial(u/P)}{\partial n} = 0, \; x \in \partial\Omega \end{cases} \qquad (2.5)$$

Then $|u(t,x) - u^*(x)| \to 0$ uniformly in $x \in \overline{\Omega}$ as $t \to \infty$, where $u^*(x)$ is the unique positive solution of (2.1).

Proof. Integrating (2.1) over Ω and using the boundary conditions in (2.5), we obtain

$$\int_\Omega u^*(x)(K(x) - u^*(x)) \, dx = 0$$

$$\Rightarrow \int_\Omega K(x)(K(x) - u^*(x)) \, dx = \int_\Omega (K(x) - u^*(x))^2 \, dx > 0, \text{ while } K(x) \not\equiv u^*(x).$$

Since $K(x) > 0$ and $u^*(x) > 0$, it concludes

$$\int_\Omega K(x)(K(x) - u^*(x)) dx > 0.$$

$$\square$$

In a similar manner, the function $v^*(x)$ is the solution of the following initial-boundary value problem and it corresponds to the survival of only species v in (1.1):

$$d_2 \Delta \left(\frac{v^*(x)}{Q(x)} \right) + v^*(x)(K(x) - v^*(x)) = 0, \ x \in \Omega, \ \frac{\partial(v^*/Q)}{\partial n} = 0, \ x \in \partial\Omega \qquad (2.6)$$

Proposition 3. *Suppose that $Q(x) \not\equiv const$, $K(x) \not\equiv const$, and $Q(x)$, $K(x)$ are linearly independent and let $v^*(x)$ be a positive solution of (2.6) then*

$$\int\limits_\Omega Q(x)(v^*(x) - K(x)) \, dx = d_2 \int\limits_\Omega \frac{|\nabla(v^*/Q)|^2}{(v^*/Q)^2} \, dx > 0. \qquad (2.7)$$

The proof of this result can be constructed similar to proposition 1.

The following inequality is also valid for (2.6):

$$\int\limits_\Omega K(x)(K(x) - v^*(x)) \, dx > 0, \ \text{unless } v^* \equiv K. \qquad (2.8)$$

Proposition 4. *The trivial equilibrium $(0,0)$ of (1.1) is unstable.*

The proof is available in [3, 12].

3   Global Stability Analysis

If the system is dynamically monotone, the best conception is to analyze the model globally instead of local analysis. Also we remark that the system (1.1) represents the interaction between two species in a wide area of both interspecific and intraspecific competitions other than to just count only two particular organisms. Taking into account these points, we consider various relations among three smooth functions $P(x)$, $Q(x)$ and $K(x)$ for further study.

3.1   Combined Effects of Spatial Functions

In this section, we want to consider different linear combinations of spatial functions and to show their heterogeneity effects.

Lemma 1. *Suppose that $P(x) \not\equiv const$, $Q(x) \not\equiv const$, $K(x) \not\equiv const$ and they are linearly independent. If $P(x) \equiv \alpha K + \beta Q$, $\alpha > 0$, $\beta > 0$ with $\alpha K, \beta Q < P$ in some non-empty open domain, and $d = d_1 = d_2$ then the semi-trivial steady state $(0, v^*(x))$ of (1.1) is unstable.*

Proof. The associated eigenvalue problem of the first equation of (1.1) around $(0, v^*(x))$ with usual boundary conditions is given by

$$d\Delta \left(\frac{\phi(x)}{P(x)} \right) + \phi(x)(K(x) - v^*(x)) = \sigma\phi(x), \ x \in \Omega, \ \frac{\partial(\phi/P)}{\partial n} = 0, \ x \in \partial\Omega \qquad (3.1)$$

The principal eigenvalue of (3.1) is given by

$$\sigma_1 \int\limits_\Omega \frac{\phi^2}{P} \, dx = \sup_{\phi \neq 0, \phi \in W^{1,2}} \left[-d \int\limits_\Omega |\nabla(\phi/P)|^2 \, dx + \int\limits_\Omega \frac{\phi^2}{P}(K(x) - v^*(x)) \, dx \right]$$

Choosing the eigenfunction $\phi(x) = P(x)$, we obtain

$$\sigma_1 \int_\Omega P(x)\,dx \geq \int_\Omega P(x)(K(x) - v^*(x))\,dx$$

and the combination $P(x) \equiv \alpha K + \beta Q$ outruns

$$\sigma_1 \int_\Omega P(x)\,dx \geq \int_\Omega (\alpha K(x) + \beta Q(x))(K(x) - v^*(x))\,dx$$

$$= \alpha \int_\Omega K(x)(K(x) - v^*(x))\,dx + \beta \int_\Omega Q(x)K(x)(1 - \frac{v^*(x)}{K(x)})\,dx$$

$$= \alpha \int_\Omega K(x)(K(x) - v^*(x))\,dx + \beta \int_\Omega Q(x)K(x)(1 - \frac{\alpha v^*(x)}{P(x) - \beta Q(x)})\,dx$$

$$> \alpha \int_\Omega K(x)(K(x) - v^*(x))\,dx + \beta \int_\Omega Q(x)K(x)(1 + \frac{\alpha v^*(x)}{\beta Q(x)})\,dx$$

where $P - \beta Q > -\beta Q$ for positive $P(x)$, $Q(x)$ and $\beta > 0$. The last integral is positive and therefore, the principal eigenvalue, $\sigma_1 > 0$ using (2.8). $\qquad\square$

Lemma 2. *Suppose that $P(x) \not\equiv const$, $Q(x) \not\equiv const$, $K(x) \not\equiv const$ and they are linearly independent. If $P(x) \equiv \alpha K + \beta Q$, $\alpha > 0$, $\beta > 0$ with αK, $\beta Q < P$ in some non-empty open domain, and $d = d_1 = d_2$ then the system (1.1) has no coexistence solution (u_s, v_s).*

Proof. Assume to the contrary that there exists a strictly positive solution $(u_s(x), v_s(x))$ of (1.1) and the system can be written as

$$\begin{cases} d\Delta\left(\dfrac{u_s(x)}{P(x)}\right) + u_s(x)(K(x) - u_s(x) - v_s(x)) = 0, \ x \in \Omega, \\[2mm] d\Delta\left(\dfrac{v_s(x)}{Q(x)}\right) + v_s(x)(K(x) - u_s(x) - v_s(x)) = 0, \ x \in \Omega, \\[2mm] \dfrac{\partial(u_s/P)}{\partial n} = \dfrac{\partial(v_s/Q)}{\partial n} = 0, \ x \in \partial\Omega. \end{cases} \qquad (3.2)$$

The following relation is constructed from (1.1) after few steps

$$\int_\Omega K(x)(K(x) - u_s(x) - v_s(x))\,dx = \int_\Omega (K(x) - u_s(x) - v_s(x))^2\,dx > 0 \qquad (3.3)$$

unless $u_s + v_s \equiv K$. Thus we have two cases
Case 1: $u_s + v_s \equiv K$

For $u_s + v_s \equiv K$, by the Maximum Principle [13], $w_s = const$ and $h_s = const$ in (1.1), where $u_s/P = w_s$ and $v_s/Q = h_s$. Therefore

$$Pw_s + Qh_s \equiv K \equiv P(x)/\alpha - (\beta/\alpha)Q(x).$$

and which implies that $w_s = 1/\alpha$ and $h_s = -(\beta/\alpha)$, a contradiction of $v_s > 0$.
Case 2: $u_s + v_s \not\equiv K$

The principal eigenvalue σ_1 of the associated eigenvalue problem of the first equation of (1.1) is defined by

$$\sigma_1 \int_\Omega \frac{\phi^2}{P}\,dx = \sup_{\phi \neq 0, \phi \in W^{1,2}} \left[-d\int_\Omega |\nabla(\phi/P)|^2\,dx + \int_\Omega \frac{\phi^2}{P}(K(x) - u_s(x) - v_s(x))\,dx \right] \qquad (3.4)$$

and we obtain $\sigma_1 \int_\Omega P(x)\,dx \geq \int_\Omega P(x)(K - u_s - v_s)dx$ by substituting $\phi(x) = P(x)$. Thus for $P(x) \equiv \alpha K + \beta Q$, we have

$$\sigma_1 \int_\Omega \alpha P(x)\,dx \geq \int_\Omega (P(x) - \beta Q(x))(K - u_s - v_s)\,dx + \beta \int_\Omega Q(x)(K - u_s - v_s)\,dx$$

$$= \alpha \int_\Omega K(x)(K - u_s - v_s)\,dx + \beta \int_\Omega Q(x)K(x)\left(1 - \frac{u_s + v_s}{K}\right)$$

$$= \alpha \int_\Omega K(x)(K - u_s - v_s)\,dx + \beta \int_\Omega Q(x)K(x)\left(1 - \frac{u_s + v_s}{P - \beta Q}\right)$$

$$> \alpha \int_\Omega K(x)(K - u_s - v_s)\,dx + \beta \int_\Omega Q(x)K(x)\left(1 + \frac{u_s + v_s}{\beta Q}\right)$$

The integral $\int_\Omega K(x)(K - u_s - v_s)\,dx > 0$ by (3.3). For the last integral, sequentially we consider $P(x) \equiv \alpha K(x) + \beta Q(x) > 0$ and $P(x) - \beta Q(x) > -\beta Q$ for positive $Q(x)$ to ensure the positivity of σ_1. Hence the zero principal eigenvalue of (3.4) contradicts, $\sigma_1 > 0$. So, there is no coexistence. $\square$

Lemma 1, Lemma 2 and Proposition 4 due to the following result in Theorem 1.

Theorem 1. *Suppose that $P(x) \not\equiv$ const, $Q(x) \not\equiv$ const, $K(x) \not\equiv$ const and they are linearly idependent. If $P(x) \equiv \alpha K + \beta Q$, $\alpha > 0$, $\beta > 0$ with αK, $\beta Q < P$ in some non-empty open domain, and $d = d_1 = d_2$ then the semi-trivial equilibrium $(u^*(x), 0)$ of (1.1) is globally asymptotically stable.*

In a similar manner, if $Q(x) \equiv \alpha K + \beta P$, $\alpha > 0$, $\beta > 0$ with non-constant $Q(x)$, $K(x)$, and $P(x)$, we have the following statement and the proof of Theorem 2 is left to the reader.

Theorem 2. *Suppose that $P(x) \not\equiv$ const, $Q(x) \not\equiv$ const, $K(x) \not\equiv$ const and they are linearly independent. If $Q(x) \equiv \alpha K + \beta P$, $\alpha > 0$, $\beta > 0$ with αK, $\beta P < Q$ in some non-empty open domain, and $d = d_1 = d_2$ then the semi-trivial equilibrium $(0, v^*(x))$ of (1.1) is globally asymptotically stable.*

It is also noted that the following result was proven in [3].

Lemma 3. *[3] Suppose that $P(x)$, $Q(x)$ and $K(x)$ are non-constant and they are linearly independent. If $K(x) \equiv \alpha P(x) + \beta Q(x)$, $\alpha > 0$, $\beta > 0$, and $d = d_1 = d_2$ then the coexistence solution $(u^*(x), v^*(x)) \equiv (\alpha P, \beta Q)$ of (1.1) is stable and the solution is unique.*

3.2 Arbitrary Distribution Functions

The main study in this section is to show the effect of distribution functions and the carrying capacity while the selection is random but bounded. First, let us assume the arbitrary function $Q(x)$ is constant; biologically it means that in system (1.1), the second species is in random walk while the species u is moving along the resource function and we want to justify the next result.

Theorem 3. *Assume that functions $P(x)$, $Q(x)$ and $K(x)$ are arbitrary, $K(x) > (P(x) + Q)$ for any $x \in \Omega$ and $Q \equiv$ const. Then the semi-trivial equilibrium $(u^*, 0)$ of (1.1) is globally asymptotically stable.*

Proof. For monotone dynamical system (1.1), it is enough to establish that two steady states $(0, v^*(x))$ and (u_s, v_s) are instable. For simplicity, we assume that $K(x) = P(x) + c^*$, where

$c^* = Q + c = const$ since $Q = const$ is our primary choice. To show the instability of $(0, v^*(x))$, consider the eigenvalue problem of (1.1) about $(0, v^*(x))$ and we get

$$\sigma_1 \int\limits_\Omega \frac{\phi^2}{P}\, dx = \sup_{\phi \neq 0, \phi \in W^{1,2}} \left[-d_1 \int\limits_\Omega |\nabla(\phi/P)|^2\, dx + \int\limits_\Omega \frac{\phi^2}{P}(K(x) - v^*(x))\, dx \right]$$

By considering the eigenfunction $\phi(x) = P(x)$, we obtain

$$\sigma_1 \int\limits_\Omega P(x)\, dx \geq \int\limits_\Omega P(x)(K(x) - v^*(x))\, dx$$

Therefore if $K(x) \equiv P(x) + c^*$ in some non-empty open domain then

$$\sigma_1 \int\limits_\Omega P(x)\, dx \geq \int\limits_\Omega (K(x) - c^*)(K(x) - v^*(x))\, dx$$

$$= c^* \int\limits_\Omega (v^*(x) - K(x))\, dx + \int\limits_\Omega K(x)(K(x) - v^*(x))\, dx$$

If $Q = const$ then from Proposition 3, we have $\int\limits_\Omega (v^*(x) - K(x))\, dx > 0$ and the rest integral is positive by (2.8). Thus σ_1 is strictly positive and $(0, v^*)$ is unstable.

The next step is to prove that there is no coexistence solution of (1.1). The following integral is driving from (1.1) for stationary solution (u_s, v_s) such that

$$\int\limits_\Omega K(x)(K(x) - u_s(x) - v_s(x))\, dx = \int\limits_\Omega (K(x) - u_s(x) - v_s(x))^2\, dx > 0 \qquad (3.5)$$

unless $u_s + v_s \equiv K$. Also for stationary solution $(u_s(x), v_s(x))$ of (1.1), we constructed

$$d_2 \Delta \left(\frac{v_s}{Q} \right) + v_s(x)(K(x) - u_s(x) - v_s(x)) = 0, x \in \Omega, \frac{\partial(v_s/Q)}{\partial n} = 0, x \in \partial\Omega$$

Dividing the first part of the above equation by v_s and then integrating over the domain and using the boundary conditions with fact $Q = const$ yields

$$\frac{d_2}{Q} \int\limits_\Omega \frac{\Delta v_s}{v_s}\, dx + \int\limits_\Omega (K(x) - u_s(x) - v_s(x))\, dx = 0$$

$$\Rightarrow \frac{d_2}{Q} \int\limits_\Omega \frac{|\nabla v_s|^2}{v_s^2}\, dx + \int\limits_\Omega (K(x) - u_s(x) - v_s(x))\, dx = 0$$

which employed the integral as defined by

$$\int\limits_\Omega (u_s(x) + v_s(x) - K(x))\, dx = \frac{d_2}{Q} \int\limits_\Omega \frac{|\nabla v_s|^2}{v_s^2}\, dx > 0 \qquad (3.6)$$

Thus we consider the following two cases from (3.5):

Case 1. For $u_s + v_s \equiv K$, by the Maximum Principle [13], $w_s = const$ and $h_s = const$ in (1.1), where $u_s/P = w_s$ and $v_s/Q = h_s$. Therefore $P(x)w_s + Qh_s \equiv K(x) \equiv P(x) + c^*$ indicates that $w_s = 1$ and $v_s = h(x) \not\equiv const$, a contradiction.

Case 2. If $u_s + v_s \not\equiv K$ then we define the eigenvalue problem

$$d_1 \Delta \left(\frac{\phi(x)}{P(x)} \right) + \phi(x)(K(x) - u_s(x) - v_s(x)) = \sigma \phi(x), \ x \in \Omega, \frac{\partial(\phi/P)}{\partial n} = 0, \ x \in \partial\Omega \qquad (3.7)$$

and the principal eigenvalue is

$$\sigma_1 \int\limits_\Omega \frac{\phi^2}{P} \, dx = \sup_{\phi \neq 0, \phi \in W^{1,2}} \left[-d_1 \int\limits_\Omega |\nabla(\phi/P)|^2 \, dx + \int\limits_\Omega \frac{\phi^2}{P}(K(x) - u_s(x) - v_s(x)) \, dx \right]$$

Choosing the eigenfunction $\phi(x) = P(x)$, the principal eigenvalue σ_1 is given by $\sigma_1 \int\limits_\Omega P(x) \, dx \geq$
$\int\limits_\Omega P(x)(K(x) - u_s(x) - v_s(x)) \, dx$. If $K(x) \equiv P(x) + c^*$ for all x over Ω then

$$\sigma_1 \int\limits_\Omega P(x) \, dx \geq \int\limits_\Omega (K(x) - c^*)(K(x) - u_s(x) - v_s(x)) \, dx$$

$$= c^* \int\limits_\Omega (u_s + v_s - K) \, dx + \int\limits_\Omega K(x)(K - u_s - v_s) \, dx > 0$$

using (3.6) and by (3.5). The zero principal eigenvalue of (3.7) contradicts $\sigma_1 > 0$ and the result is justified. $\qquad\square$

Theorem 4. *Assume that functions $P(x), Q(x)$ and $K(x)$ are arbitrary and $K(x) > (P(x) + Q(x))$ for any $x \in \Omega$. If there exists some $Q(x)$ such that $\int\limits_\Omega \frac{\nabla(v^*/Q)\nabla v^*}{(v^*)^2} \, dx \geq 0$, the semi-trivial equilibrium $(u^*, 0)$ of (1.1) is globally asymptotically stable.*

Proof. For monotonic dynamical system (1.1), we have to establish that the equilibrium $(0, v^*)$ is unstable and there is no co-existence solution.

Let us now turn the inequality $K(x) > (P(x) + Q(x))$ to $K(x) \equiv P(x) + Q(x) + c$ for $c > 0$. To show the instability of $(0, v^*)$, consider the first equation of (1.1) about $(0, v^*)$ and the principal eigenvalue is defined as

$$\sigma_1 \int\limits_\Omega \frac{\phi^2}{P} \, dx = \sup_{\phi \neq 0, \phi \in W^{1,2}} \left[-d_1 \int\limits_\Omega |\nabla(\phi/P)|^2 \, dx + \int\limits_\Omega \frac{\phi^2}{P}(K - v^*) \, dx \right]$$

Choosing the eigenfunction $\phi(x) = P(x)$ using the connection $K(x) \equiv P(x) + Q(x) + c$, we get

$$\sigma_1 \int\limits_\Omega P(x) \, dx \geq \int\limits_\Omega P(x)(K - v^*) \, dx$$

$$= \int\limits_\Omega (K(x) - Q(x) - c)(K - v^*) \, dx$$

$$= \int\limits_\Omega K(x)(K - v^*) \, dx + \int\limits_\Omega Q(x)(v^* - K) \, dx + c \int\limits_\Omega (v^* - K) \, dx \qquad (3.8)$$

the first two integrals of right hand side of (3.8) is positive by Proposition 3.

To check the positivity of $\int\limits_\Omega (v^* - K) \, dx$, let us now explore some additional investigations: Since $v^* > 0$, dividing both sides of (2.6) by v^*, and then integrating over Ω yields

$$d_2 \int\limits_\Omega \frac{\Delta(v^*/Q)}{v^*} \, dx + \int\limits_\Omega (K - v^*) \, dx = 0$$

Employing the respective boundary conditions, the outcome is

$$\int\limits_{\Omega} (v^* - K)\, dx = \int\limits_{\Omega} \frac{\nabla(v^*/Q)\nabla v^*}{(v^*)^2}\, dx \geq 0$$

as long as $\int\limits_{\Omega} \frac{\nabla(v^*/Q)\nabla v^*}{(v^*)^2}\, dx \geq 0$. Therefor the principal eigenvalue is positive and so $(0, v^*)$ is unstable.

Next step is to prove that there is no co-existence solution of (1.1). Continuing the previous techniques, it is easy to construct the following two integral relations from (1.1)

$$\int\limits_{\Omega} K(x)(K - u_s - v_s)\, dx = \int\limits_{\Omega} (K - u_s - v_s)^2\, dx > 0, \text{ unless } u_s + v_s \equiv K \qquad (3.9)$$

and

$$\int\limits_{\Omega} Q(x)(u_s + v_s - K)\, dx = d_2 \int\limits_{\Omega} \frac{|\nabla(v_s/Q)|^2}{(v_s/Q)^2}\, dx > 0 \qquad (3.10)$$

By taking into account $u_s + v_s = K$ and applying the maximum principal [13], $w_s = const > 0$ and $h_s = const > 0$ in (1.1), where $u_s/P = w_s$ and $v_s/Q = h_s$. Therefore

$$P(x)w_s + Q(x)h_s \equiv K(x) \equiv P(x) + Q(x) + c \qquad (3.11)$$

In (3.11), if $w_s = 1$ it is seen that $Qh_s = Q + c$, which implies $Q = \frac{c}{h_s - 1} = const$, which is a contradiction. However if we choose $h_s = 1$ then we obtain $Pw_s = P + c$, which implies $P = \frac{c}{w_s - 1} = const$, again a contradiction.

Instead of $u_s + v_s \equiv K$, if $u_s + v_s \not\equiv K$ then the principal eigenvalue analysis import

$$\sigma_1 \int\limits_{\Omega} P(x)\, dx \geq \int\limits_{\Omega} P(x)(K - u_s - v_s)\, dx$$
$$= \int\limits_{\Omega} K(x)(K - u_s - v_s)\, dx + \int\limits_{\Omega} Q(x)(u_s + v_s - K)\, dx + c\int\limits_{\Omega} (u_s + v_s - K)\, dx$$

Immediate first two integrals of right side are positive by (3.9) and (3.10) and the last one is

$$\int\limits_{\Omega} (u_s + v_s - K)\, dx = \int\limits_{\Omega} \frac{\nabla(v_s/Q)\nabla v_s}{(v_s)^2}\, dx \geq 0$$

by the introductory assumption. So, $\sigma_1 > 0$ and there is no co-existence solution, which concludes the proof. $\qquad\square$

Remark 1. *Since there are many choice of $Q(x)$, it is remarked that Theorem 4 is not always stable if $\int\limits_{\Omega} \frac{\nabla(v_s/Q)\nabla v_s}{(v_s)^2}\, dx < 0$. On that point, possible outcomes are the co-existence of both species and the extinction of one by other. By illustrating numerical examples, we will investigate these situations in next section.*

4 Numerical Illustrations

In the following set of examples, we study the role played by initial densities of two species, various linear combinations and the arbitrary relations between two spatial distributions $P(x)$, $Q(x)$ and the carrying capacity $K(x)$.

Example 3. *Consider $P(x) = 2.1 + \cos(\pi x)$, $Q(x) = 1.4 + \cos(\pi x)$ and $K(x) \equiv P(x) + Q(x)$ on $\Omega = (0,1)$ with equal diffusion coefficients. Consider initial condition $u(0,x) = 0.8$, $v(0,x) = 1.8$,*
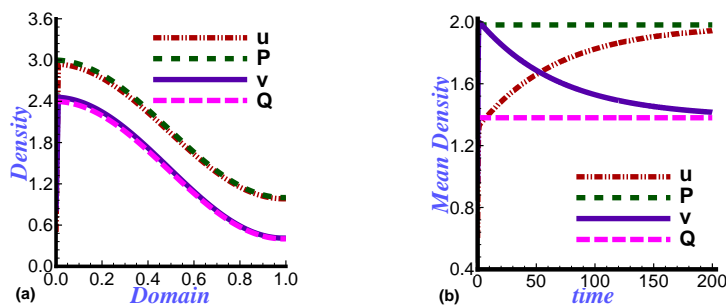
Fig. 3. Solutions of (1.1) at $t = 200$ in (a) and average solutions of (1.1) in (b) for $P(x) = 2.1 + \cos(\pi x)$, $Q(x) = 1.4 + \cos(\pi x)$, and $K(x) \equiv P(x) + Q(x)$, $d_1 = d_2 = 1.0$, on $\Omega = (0,1)$, with initial densities $(u_0, v_0) = (0.8, 1.8)$.

then $P(x)$ and $Q(x)$ are the unique exact stationary solution of (1.1) and the coexistence is globally attractive as observed in both (a) and (b) of Figure 3. It is remarked that when $K(x) \equiv \alpha P(x) + \beta Q(x)$ for any $x \in \Omega$ then the theory for this type of result was proven in [3].

Example 4. *Let us now assume that $K(x) = 2.0 + \cos(\pi x)$, $Q(x) = 1.4 + \cos(\pi x)$ and $P(x) \equiv K(x) + Q(x) = 3.4 + 2\cos(\pi x)$ on $\Omega = (0,1)$ with identical diffusion coefficients. Then for non-*
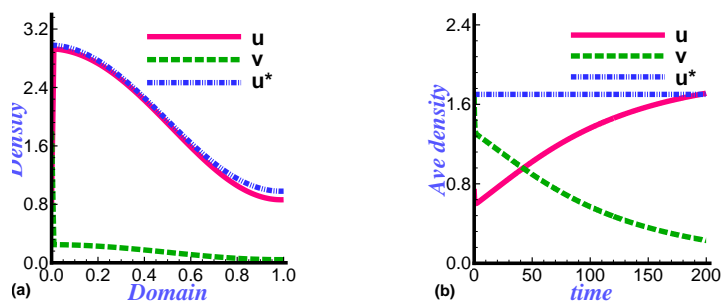
Fig. 4. Solutions of (1.1) at $t = 200$ in (a) and average solutions of (1.1) in (b) for $K(x) = 2.0 + \cos(\pi x)$, $Q(x) = 1.4 + \cos(\pi x)$, and $P(x) \equiv K(x) + Q(x)$, $d_1 = d_2 = 1.0$, on $\Omega = (0,1)$, with $(u_0, v_0) = (0.8, 1.8)$.

negative $u_0 = 0.8$, $v_0 = 1.8$, the equilibrium $(u^*, 0)$ is globally asymptotically stable and it ensure the theoretical result described and justified in Theorem 1, here in particular $\alpha = \beta = 1.0$. Check the well designed results which are depicted in both diagrams of Figure 4. In the biological point of view, the species u is consuming more resources in competition with v and as seen $P(x) > Q(x)$ over the habitat.

Example 5. *Consider $K(x) = 2.2 + \cos(\pi x)$, $P(x) = 1.3 + \cos(\pi x)$ and $Q(x) \equiv K(x) + P(x)$ on $\Omega = (0,1)$ with equal diffusion coefficients. We assume the initial population sizes are $u_0 =$*
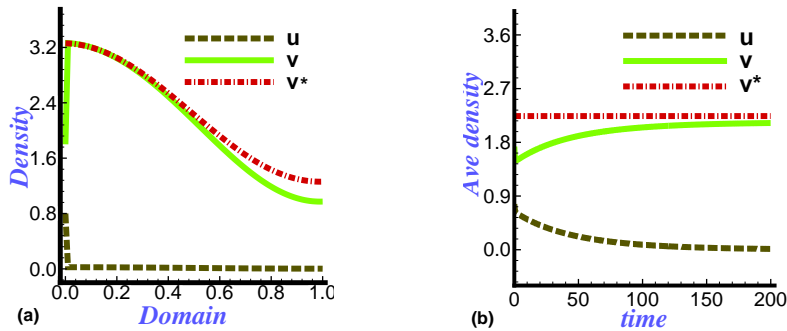
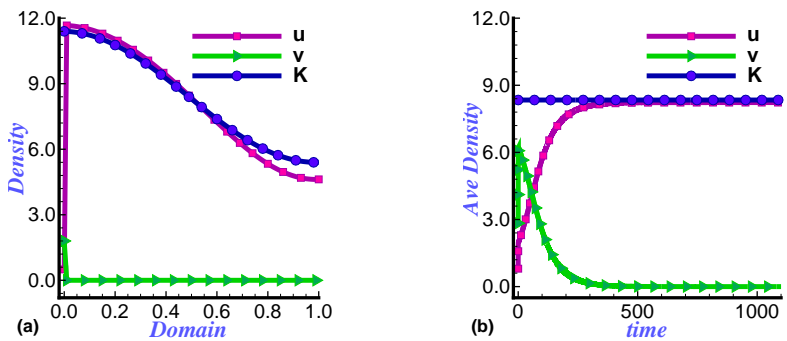

Fig. 5. Solutions of (1.1) at $t = 200$ in (a), and average solutions of (1.1) in (b) for
$K(x) = 2.2 + \cos(\pi x)$, $P(x) = 1.3 + \cos(\pi x)$, **and** $Q(x) \equiv K(x) + P(x)$, $d_1 = d_2 = 1.0$, **on**
$\Omega = (0,1)$, **with** $(u_0, v_0) = (0.8, 1.8)$.

0.8, $v_0 = 1.8$.

Then the semi-trivial equilibrium $(0, v^*)$ of (1.1) is globally attractive independently of diffusion speeds and the result is verified in Theorem 2. See the numerical results as displayed in left and right Figures of 5.

Example 6. *Let us now consider the following arbitrary functions $P(x) \equiv 2.0 + \cos(\pi x)$, $Q(x) \equiv 3.4 + 2\cos(\pi x)$ and $K(x) \equiv P(x) + Q(x) + 3.0$ such that $K(x) > P(x) + Q(x)$ in a non-empty open*

Fig. 6. Solutions of (1.1) at $t = 2000$ in (a), and average solutions of (1.1) in terms of
time in (b) for $K(x) \equiv P(x) + Q(x) + 3.0$ where $P(x) \equiv 2.0 + \cos(\pi x)$, $Q(x) \equiv 3.4 + 2\cos(\pi x)$,
and $d_1 = d_2 = 1.0$, on $\Omega = (0,1)$, with $(u_0, v_0) = (0.5, 1.8)$.

sub-domain $\Omega_s \subset \Omega$ with initial values $u_0 = 0.5$, $v_0 = 1.8$ and identical dispersal rates $d_1 = d_2 = 1.0$. Thus for non-negative and non-trivial (u_0, v_0) with a random choice of $Q(x)$, the equilibrium $(u^*, 0)$ is globally asymptotically stable as depicted in two figures of 6. The numerical illustrations correspond the analytical hypothesis as verified in Theorem 4.

5 Conclusion

In this paper, we studied a two species diffusive model, where both species are adopted according to two individuals directed functions. While $P(x), Q(x)$ and $K(x)$ are linearly independent, then for the strict inequality $\alpha K(x) < P(x)$, $\beta Q(x) < P(x)$, the coexistence solution is not stable and in competition, one species repelled. Here the semi-trivial solution $(u^*(x), 0)$ is globally asymptotically stable. For random choice of $K(x)$, $P(x)$, and $Q(x)$, it is proven that $(u^*(x), 0)$ is globally asymptotically stable for estimated non-negative integrand depending on $Q(x)$; see Fig. 6. All these functions are spatially periodic, bounded and strictly positive. If the distribution function $P(x)$ is equivalence to $Q(x)$ and non-proportional to $K(x)$ then for the same diffusion coefficient, the non-trivial equilibria are locally asymptotically stable, see Examples 2. Some numerical examples were presented to justify the analytic study in a non-empty open domain with various initial conditions. This problem defined in (1.1) is biologically meaningful due to the activities of various species; specially for grazing animals, birds, marine organisms. Someone might have interest to extend these results for periodic (both space-time dependent) smooth distribution functions and carrying capacity. Finally, we have some open problems to the readers for further analysis:

1. Introduce different boundary conditions in (1.1) and study the problem.

2. Considering non-equivalent functional carrying capacities for both species, show that population with higher carrying capacity leads to the rest one is in extinction.

3. It will be interesting if we add the harvesting term for both populations in (1.1) and study the revised model.

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Competing Interests

Authors have declared that no competing interests exist.

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