Hindawi
Advances in Mathematical Physics
Volume 2022, Article ID 3847889, 8 pages
https://doi.org/10.1155/2022/3847889

Research Article

Conformal η-Ricci-Yamabe Solitons within the Framework of ϵ-LP-Sasakian 3-Manifolds

Abdul Haseeb [1] **and Meraj Ali Khan** [2]

[1]*Department of Mathematics, Faculty of Science, Jazan University, Jazan-45142, Saudi Arabia*
[2]*Department of Mathematics, University of Tabuk, Tabuk, Saudi Arabia*

Correspondence should be addressed to Meraj Ali Khan; meraj79@gmail.com

Received 23 May 2022; Accepted 28 June 2022; Published 2 August 2022

Academic Editor: Antonio Scarfone

In the present note, we study ϵ-LP-Sasakian 3-manifolds $M^3(\epsilon)$ whose metrics are conformal η-Ricci-Yamabe solitons (in short, CERYS), and it is proven that if an $M^3(\epsilon)$ with a constant scalar curvature admits a CERYS, then $\pounds_U\zeta$ is orthogonal to ζ if and only if $\Lambda - \epsilon\sigma = -2\epsilon l + (mr/2) + (1/2)(p + (2/3))$. Further, we study gradient CERYS in $M^3(\epsilon)$ and proved that an $M^3(\epsilon)$ admitting gradient CERYS is a generalized conformal η-Einstein manifold; moreover, the gradient of the potential function is pointwise collinear with the Reeb vector field ζ. Finally, the existence of CERYS in an $M^3(\epsilon)$ has been drawn by a concrete example.

1. Introduction

The index of a metric generates variety of vector fields such as space-like, time-like, and light-like vector fields. Therefore, the study of manifolds with indefinite metrics becomes of great importance in physics and relativity. About three decades ago, the concept of ϵ-Sasakian manifolds was intoduced by Bejancu and Duggal [1]. Later, Xufeng and Xiaoli [2] have shown that these manifolds are real hypersurfaces of indefinite Kaehlerian manifolds. Recently, the manifolds with indefinite structures have also been studied by several authors such as [3–7].

The concept of conformal Ricci flow was introduced by Fischer [8] as a generalization of the classical Ricci flow equation, which is defined on an n-dimensional Riemannian manifold M by the equations

$$\frac{\partial g}{\partial t} = -2\left(S + \frac{g}{n}\right) - pg, \quad r(g) = -1, \tag{1}$$

where p defines a time dependent nondynamical scalar field (also called the conformal pressure), g is the Riemannian metric, and r and S represent the scalar curvature and the Ricci tensor of M, respectively. The term $-pg$ plays a role of constraint force to maintain r in the above equation.

In 2015, Basu and Bhattacharya [9] proposed the concept of conformal Ricci soliton on M and is defined by

$$\pounds_U g + 2S = \left\{\frac{1}{n}(pn + 2) - 2\Lambda\right\} g, \tag{2}$$

where $\pounds_U$ represents the Lie derivative operator along the smooth vector field U on M and $\Lambda \in \mathbb{R}$ ($\mathbb{R}$ is the set of real numbers).

In [10], Guler and Crasmareanu established a scalar combination of Ricci and Yamabe flows; this new class of geometric flows called Ricci-Yamabe flow of type (l, m) and is defined by

$$\frac{\partial}{\partial t} g(t) = 2lS(g(t)) - mr(t)g(t), \quad g(0) = g_0, \tag{3}$$

for some scalars l and m.

A solution to the Ricci-Yamabe flow is called Ricci-Yamabe soliton if it depends only on one parameter group of diffeomorphism and scaling. A Riemannian manifold is said to have a Ricci-Yamabe solitons (RYS) if [11]

$$\pounds_U g + (2\Lambda - mr)g + 2lS = 0, \quad (4)$$

where $l, m, \Lambda \in \mathbb{R}$.

In [12], Zhang et al. studied conformal Ricci-Yamabe soliton (CRYS), which is defined on an n-dimensional Riemannian manifold by

$$\pounds_U g + 2lS + \left\{2\Lambda - mr - \frac{1}{n}(pn+2)\right\}g = 0. \quad (5)$$

Motivated by the above studies, we introduce the notion of conformal η-Ricci-Yamabe soliton (CERYS). A Riemannian manifold M of dimension n is said to have CERYS if

$$\pounds_U g + 2lS + \left\{2\Lambda - mr - \frac{1}{n}(pn+2)\right\}g + 2\sigma\,\eta\otimes\eta = 0, \quad (6)$$

where $l, m, \Lambda, \sigma \in \mathbb{R}$ and η is a 1-form on M.

If U is the gradient of a smooth function f on M, then equation (6) is called the gradient conformal η-Ricci-Yamabe soliton (gradient CERYS) and takes the form

$$\nabla^2 f + lS + \left\{\Lambda - \frac{mr}{2} - \frac{1}{2}\left(p + \frac{2}{n}\right)\right\}g + \sigma\,\eta\otimes\eta = 0, \quad (7)$$

where $\nabla^2 f$ is said to be the Hessian of f. A CRYS (or gradient CRYS) is said to be shrinking, steady or expanding if $\Lambda < 0$, $= 0$ or >0, respectively. A CERYS (or gradient CERYS) reduces to

(i) conformal η – Ricci soliton if $l = 1, m = 0$,

(ii) conformal η – Yamabe soliton if $l = 0, m = 1$,

(iii) conformal η – Einstein soliton if $l = 1, m = -1$.

If $S(V_1, V_2) = \{\Lambda - (mr/2) - (1/2)(p + (2/n))\}g(V_1, V_2) + \sigma\,\eta(V_1)\eta(V_2)$ for all vector fields V_1, V_2 on M, then we call the manifold as a conformal η-Einstein manifold. Further, if $\sigma = 0$, that is, $S(V_1, V_2) = \{\Lambda - (mr/2) - (1/2)(p + (2/n))\}g(V_1, V_2)$, then M is called a conformal Einstein manifold. If an ϵ-LP-Sasakian 3-manifold $M^3(\epsilon)$ satisfies (6) (resp., (7)), then we say that $M^3(\epsilon)$ admits a CERYS (resp., gradient CERYS).

The study of indefinite structures of the manifolds admitting various types of solitons is of high interest of researchers from different fields due to its wide applications in general relativity, cosmology, quantum field theory, string theory, thermodynamics, etc. This is why, the researchers from various fields are attracted by this study. For more details about the related studies, we recommend the papers ([13–25]) and the references therein.

In this paper, we handle the study of $M^3(\epsilon)$ admitting CERYS. The article is unfolded as follows: Preliminaries on $M^3(\epsilon)$ are the focus of Section 2. Sections 3 and 4 are dedicated to conferring the CERYS and gradient CERYS in $M^3(\epsilon)$, respectively. At last, we model an example of $M^3(\epsilon)$ which helps to examine the existence of CERYS on $M^3(\epsilon)$.

2. Preliminaries

A differentiable manifold of dimension n is called an ϵ-Lorentzian para-Sasakian (in short, $M^3(\epsilon)$), in case it admits a $(1,1)$ tensor field φ, a contravariant vector field ζ, a 1-form η, and a Lorentzian metric g fulfilling [6]

$$\varphi^2 V_1 = V_1 + \eta(V_1)\zeta, \; \eta(\zeta) = -1, \quad (8)$$

$$g(\zeta,\zeta) = -\epsilon, \quad \eta(V_1) = \epsilon g(V_1,\zeta), \quad \varphi\zeta = 0, \quad \eta(\varphi V_1) = 0, \quad (9)$$

$$g(\varphi V_1, \varphi V_2) = g(V_1, V_2) - \epsilon\eta(V_1)\eta(V_2), \quad (10)$$

$$(\nabla_{V_1}\varphi)V_2 = g(V_1, V_2)\zeta + \epsilon\eta(V_2)V_1 + 2\epsilon\eta(V_1)\eta(V_2)\zeta, \quad (11)$$

$$\nabla_{V_1}\zeta = \epsilon\varphi V_1, \quad (12)$$

for all vector fields V_1, V_2 on $M^3(\epsilon)$, where ϵ is -1 or 1 according as ζ is space-like or time-like vector field, and ∇ represents the Levi-Civita connection with respect to g.

Moreover, in an $M^3(\epsilon)$, we have [6, 22]

$$(\nabla_{V_1}\eta)V_2 = \Phi(V_1, V_2) = g(\varphi V_1, V_2), \quad (13)$$

$$R(V_1, V_2)\zeta = \eta(V_2)V_1 - \eta(V_1)V_2, \quad (14)$$

$$R(\zeta, V_1)V_2 = \epsilon g(V_1, V_2)\zeta - \eta(V_2)V_1, \quad (15)$$

$$R(\zeta, V_1)\zeta = -R(V_1,\zeta)\zeta = V_1 + \eta(V_1)\zeta, \quad (16)$$

$$S(V_1,\zeta) = 2\eta(V_1) \Longleftrightarrow Q\zeta = 2\epsilon\zeta, \quad (17)$$

where Φ is a symmetric $(0,2)$ tensor field, R is the curvature tensor, and Q is the Ricci operator related by $g(QV_1, V_2) = S(V_1, V_2)$.

We note that if $\epsilon = 1$ and ζ is time-like vector field, then an $M^3(\epsilon)$ is usual LP-Sasakian manifold of dimension 3.

Definition 1. An $M^3(\epsilon)$ is called a generalized η-Einstein manifold if its Ricci tensor $S(\neq 0)$ satisfies

$$S(V_1, V_2) = ag(V_1, V_2) + b\eta(V_1)\eta(V_2) + cg(\varphi V_1, V_2), \quad (18)$$

where a, b, and c are scalar functions of ϵ. If $c = 0$ (resp., $b = c = 0$), then $M^3(\epsilon)$ is called η-Einstein (resp., Einstein) manifold.

Proposition 2. *In an $M^3(\epsilon)$, the Ricci tensor S is expressed as*

$$S(V_1, V_2) = \left(\frac{r}{2} - \epsilon\right) g(V_1, V_2) + \left(\frac{\epsilon r}{2} - 3\right) \eta(V_1)\eta(V_2), \tag{19}$$

for any V_1, V_2 on $M^3(\epsilon)$.

Proof. Since in an $M^3(\epsilon)$, the conformal curvature tensor vanishes, therefore, we have

$$\begin{aligned} R(V_1, V_2)V_3 &= S(V_2, V_3)V_1 - S(V_1, V_3)V_2 \\ &+ g(V_2, V_3)QV_1 - g(V_1, V_3)QV_2 \\ &- \frac{r}{2}(g(V_2, V_3)V_1 - g(V_1, V_3)V_2), \end{aligned} \tag{20}$$

which by putting $V_3 = \zeta$ then using (9), (14), and (17) leads to

$$\eta(V_2)QV_1 - \eta(V_1)QV_2 = \left(\epsilon - \frac{r}{2}\right)(\eta(V_1)V_2 - \eta(V_2)V_1). \tag{21}$$

Again, putting $V_2 = \zeta$ in (21) then using (8) and (17), we find

$$QV_1 = \left(\frac{r}{2} - \epsilon\right)V_1 + \left(\frac{r}{2} - 3\epsilon\right)\eta(V_1)\zeta. \tag{22}$$

The inner product of (22) with V_2 gives (19). □

3. $M^3(\epsilon)$ Admitting CERYS

First, we prove the following theorem.

Theorem 3. *If an $M^3(\epsilon)$ with the constant scalar curvature admits a CERYS, then*

$$\Lambda - \epsilon\sigma = -2\epsilon l + \frac{mr}{2} + \frac{1}{2}\left(p + \frac{2}{3}\right). \tag{23}$$

Moreover, $£_U\zeta$ is orthogonal to ζ if and only if (23) holds.

Proof. Let an $M^3(\epsilon)$ admit a CERYS, then by using (19) in (6), we have

$$\begin{aligned} (£_U g)(V_1, V_2) &= -\left\{(l - m)r + 2\Lambda - 2\epsilon l - \left(p + \frac{2}{3}\right)\right\} g(V_1, V_2) \\ &- \{l(\epsilon r - 6) + 2\sigma\}\eta(V_1)\eta(V_2). \end{aligned} \tag{24}$$

The covariant differentiation of (24) with respect to V_3 leads to

$$\begin{aligned} \left(\nabla_{V_3} £_U g\right)(V_1, V_2) &= -l(V_3 r)\, g(\varphi V_1, \varphi V_2) + m(V_3 r)g(V_1, V_2) \\ &- \{l(\epsilon r - 6) + 2\sigma\}(g(\varphi V_3, V_1)\eta(V_2) + g(\varphi V_3, V_2)\eta(V_1)). \end{aligned} \tag{25}$$

As g is parallel with respect to ∇, then the relation [26].

$$\begin{aligned} &\left(£_U \nabla_{V_1} g - \nabla_{V_1} £_U g - \nabla_{[U, V_1]} g\right)(V_2, V_3) \\ &= -g((£_U\nabla)(V_1, V_3), V_2) - g((£_U\nabla)(V_1, V_2), V_3), \end{aligned} \tag{26}$$

turns to

$$\begin{aligned} \left(\nabla_{V_1} £_U g\right)(V_2, V_3) &= g((£_U\nabla)(V_1, V_3), V_2) \\ &+ g((£_U\nabla)(V_1, V_2), V_3). \end{aligned} \tag{27}$$

Due to symmetric property of $£_U\nabla$, equation (27) takes the form

$$\begin{aligned} 2g((£_U\nabla)(V_1, V_2), V_3) &= \left(\nabla_{V_1} £_U g\right)(V_2, V_3) \\ &+ \left(\nabla_{V_2} £_U g\right)(V_1, V_3) \\ &- \left(\nabla_{V_3} £_U g\right)(V_1, V_2). \end{aligned} \tag{28}$$

Using (25) in (28), we have

$$\begin{aligned} &2g((£_U\nabla)(V_1, V_2), V_3) = \\ &-l\{(V_1 r)g(\varphi V_2, \varphi V_3) + (V_2 r)g(\varphi V_1, \varphi V_3) - (V_3 r)g(\varphi V_1, \varphi V_2)\} \\ &+ m\{(V_1 r)g(V_2, V_3) + (V_2 r)g(V_1, V_3) - (V_3 r)g(V_1, V_2)\} \\ &- 2\{l(\epsilon r - 6) + 2\sigma\}g(\varphi V_1, V_2)\eta(V_3). \end{aligned} \tag{29}$$

By eliminating V_3 from the foregoing equation, it follows that

$$\begin{aligned} &2(£_U\nabla)(V_1, V_2) = \\ &-l\{(V_1 r)(V_2 + \eta(V_2)\zeta) + (V_2 r)(V_1 + \eta(V_1)\zeta) - (Dr)g(\varphi V_1, \varphi V_2)\} \\ &- 2\epsilon\{l(\epsilon r - 6) + 2\sigma\}g(\varphi V_1, V_2)\zeta + m\{(V_1 r)V_2 + (V_2 r)V_1 - (Dr)g(V_1, V_2)\}, \end{aligned} \tag{30}$$

where $V_1 l = g(Dl, V_1)$, D stands for the gradient operator with respect to g. Taking $V_2 = \zeta$ and using r constant (hence $(Dr = 0)$ and $(\zeta r = 0)$), (30) turns to

$$(£_U\nabla)(V_1, \zeta) = 0. \tag{31}$$

The covariant derivative of (31) with respect to V_2 leads to

$$\left(\nabla_{V_2} £_U \nabla\right)(V_1, \zeta) = -\epsilon(£_U\nabla)(V_1, \varphi V_2), \tag{32}$$

which by using in $(\nabla_U R)(V_1, V_2)V_3 = (\nabla_{V_1} £_U\nabla)(V_2, V_3) - (\nabla_{V_2} £_U\nabla)(V_1, V_3)$, we deduce

$$(\nabla_U R)(V_1, \zeta)\zeta = 0. \tag{33}$$

The Lie derivative of $R(V_1, \zeta)\zeta = -V_1 - \eta(V_1)\zeta$ along U

yields

$$(\nabla_U R)(V_1,\zeta)\zeta + 2\eta(\pounds_U\zeta)V_1 - \epsilon g(V_1,\pounds_U\zeta)\zeta = -(\pounds_U\eta)(V_1)\zeta, \quad (34)$$

which by using (33) reduces to

$$(\pounds_U\eta)(V_1)\zeta = -2\eta(\pounds_U\zeta)V_1 + \epsilon g(V_1,\pounds_U\zeta)\zeta. \quad (35)$$

Now, taking the Lie derivative of $\eta(V_1)=\epsilon g(V_1,\zeta)$, it follows that

$$(\pounds_U\eta)V_1 = \epsilon(\pounds_U g)(V_1,\zeta) + \epsilon g(V_1,\pounds_U\zeta). \quad (36)$$

Taking $V_2=\zeta$ in (24), we find

$$(\pounds_U g)(V_1,\zeta) = \left\{-2\epsilon\Lambda + \epsilon mr - 4l + 2\sigma + \epsilon\left(p+\frac{2}{3}\right)\right\}\eta(V_1). \quad (37)$$

Again, taking the Lie-derivative of $g(\zeta,\zeta)=-\epsilon$, we have

$$(\pounds_U g)(\zeta,\zeta) = -2\epsilon\eta(\pounds_U\zeta). \quad (38)$$

Now, by combining the equations (35)–(38), we have

$$\left\{2\epsilon\Lambda - \epsilon mr + 4l - 2\sigma - \epsilon\left(p+\frac{2}{3}\right)\right\}\varphi^2 V_1 = 0. \quad (39)$$

From the foregoing equation, it follows that

$$\Lambda - \epsilon\sigma = -2\epsilon l + \frac{mr}{2} + \frac{1}{2}\left(p+\frac{2}{3}\right) = 0, \quad (40)$$

where $\varphi^2 V_1 \neq 0$.

Next, from the equations (37)–(40), we observe that $\eta(\pounds_U\zeta)=0$, i.e., $\pounds_U\zeta$ is orthogonal to ζ. Conversely, from (37) and (38), one can see that if $\pounds_U\zeta$ is orthogonal to ζ, then (40) immediately follows. This completes the proof. □

In particular, if $l=1, m=\sigma=0$, then (40) reduces to $\Lambda=-2\epsilon+(1/2)(p+(2/3))$. Thus, we have the following.

Corollary 4. *If an $M^3(\epsilon)$ with the constant scalar curvature admits a conformal Ricci soliton, then the soliton on $M^3(\epsilon)$ is concluded as follows:*

(i) if $\epsilon=1$, (i.e., ζ is time-like), then the soliton on $M^3(\epsilon)$ is expanding, steady, or shrinking according to $p>(10/3)$, $=(10/3)$, or $<(10/3)$

(ii) if $\epsilon=-1$, (i.e., ζ is space-like), then the soliton on $M^3(\epsilon)$ is expanding, steady or shrinking according to $p>(-14/3)$, $=(-14/3)$, or $<(-14/3)$

Next, if $m=1, l=\sigma=0$, then (40) reduces to $\Lambda=(r/2)+(1/2)(p+(2/3))$. Thus, we have the following.

Corollary 5. *If an $M^3(\epsilon)$ with the constant scalar curvature admits a conformal Yamabe soliton, then the soliton on $M^3(\epsilon)$ is expanding, steady or shrinking according to $p>-(r+(2/3))$, $=-(r+(2/3))$ or $<-(r+(2/3))$.*

Again, if $l=1, m=-1, \sigma=0$, then (40) reduces to $\Lambda=-2\epsilon-(r/2)+(1/2)(p+(2/3))$. Thus, we have the following.

Corollary 6. *If an $M^3(\epsilon)$ with the constant scalar curvature admits a conformal Einstein soliton, then the soliton on $M^3(\epsilon)$ is concluded as follows:*

(i) if $\epsilon=1$, (i.e., ζ is time-like), then the soliton on $M^3(\epsilon)$ is expanding, steady, or shrinking according to $p>(10/3)+r$, $=(10/3)+r$ or $<(10/3)+r$

(ii) if $\epsilon=-1$, (i.e., ζ is space-like), then the soliton on $M^3(\epsilon)$ is expanding, steady or shrinking according to $p>(-14/3)+r$, $=(-14/3)+r$ or $<(-14/3)+r$.

Furthermore, let an $M^3(\epsilon)$ admit a CERYS at $U=\zeta$, then from (6), we have

$$(\pounds_\zeta g)(V_1,V_2) + 2lS(V_1,V_2) + \left\{2\Lambda - mr - \left(p+\frac{2}{3}\right)\right\}g(V_1,V_2) + 2\sigma\,\eta(V_1)\eta(V_2) = 0, \quad (41)$$

which by using the value $(\pounds_\zeta g)(V_1,V_2)=g(\nabla_{V_1}\zeta,V_2)+g(V_1,\nabla_{V_2}\zeta)=2\epsilon g(\varphi V_1,V_2)$, we arrive

$$S(V_1,V_2) = -\frac{1}{l}\left\{\Lambda - \frac{mr}{2} - \frac{1}{2}\left(p+\frac{2}{3}\right)\right\}g(V_1,V_2) - \frac{\sigma}{l}\eta(V_1)\eta(V_2) - \frac{\epsilon}{l}g(\varphi V_1,V_2), \quad \text{where} \quad l\neq 0. \quad (42)$$

By putting $V_2=\zeta$ in (42) and using (17), we find

$$\Lambda - \epsilon\sigma = -2\epsilon l + \frac{mr}{2} + \frac{1}{2}\left(p+\frac{2}{3}\right). \quad (43)$$

Thus, we have the following.

Corollary 7. *If an $M^3(\epsilon)$ admits a CERYS at $U=\zeta$, then $M^3(\epsilon)$ is a generalized conformal η-Einstein manifold and the scalars Λ and σ are related by (43). Moreover, the nature of the soliton on $M^3(\epsilon)$ is concluded as Corollaries 4 and 6.*

Definition 8. A vector field U on an $M^3(\epsilon)$ is called torse forming vector field in case [27].

$$\nabla_{V_1}U = fV_1 + \gamma(V_1)U, \quad (44)$$

where f and γ are smooth function and 1-form, respectively.

Let us consider an $M^3(\epsilon)$ admitting a CERYS, further considering the Reeb vector field ζ as a torse-forming vector field. Thus, from (44), we have

$$\nabla_{V_1}\zeta = fV_1 + \gamma(V_1)\zeta, \tag{45}$$

for all V_1 on $M^3(\epsilon)$. Taking the inner product of (45) with ζ, we find

$$g\left(\nabla_{V_1}\zeta, \zeta\right) = \epsilon f\eta(V_1) - \epsilon\gamma(V_1). \tag{46}$$

Also, from (12), we find

$$g\left(\nabla_{V_1}\zeta, \zeta\right) = 0. \tag{47}$$

Thus, the last two equations give $\gamma = f\eta$ (where $\epsilon \neq 0$), and hence (45) turns to

$$\nabla_{V_1}\zeta = f(V_1 + \eta(V_1)\zeta). \tag{48}$$

Now, in view of (48), we have

$$\left(£_\zeta g\right)(V_1, V_2) = 2f\{g(V_1, V_2) + \eta(V_1)\eta(V_2)\}. \tag{49}$$

By virtue of (49), (42) turns to

$$\begin{aligned} S(V_1, V_2) = &-\frac{1}{l}\left\{\Lambda + f - \frac{mr}{2} - \frac{1}{2}\left(p + \frac{2}{3}\right)\right\}g(V_1, V_2) \\ &- \frac{1}{l}(\epsilon f + \sigma)\,\eta(V_1)\eta(V_2), l \neq 0. \end{aligned} \tag{50}$$

Thus, we state the following.

Theorem 9. *If an $M^3(\epsilon)$ admits a CERYS at $U = \zeta$ with torse-forming vector field ζ. Then, $M^3(\epsilon)$ is a conformal η-Einstein manifold.*

In particular, if $\sigma = -\epsilon f$, then (50) takes the form $S(V_1, V_2) = -(1/l)\{\Lambda + f - (mr/2) - (1/2)(p + (2/3))\}g(V_1, V_2)$, $l \neq 0$. Thus, we have the following.

Corollary 10. *An $M^3(\epsilon)$ admitting a CERYS with torse-forming vector field ζ is a conformal Einstein manifold if $\sigma = f$ for space-like vector field (or $\sigma = -f$ for time-like vector field).*

4. Gradient CERYS on $M^3(\epsilon)$

Let the metric g on $M^3(\epsilon)$ be a gradient CERYS. Then, equation (7) can be expressed as

$$\nabla_{V_2}Df + lQV_2 + \left\{\Lambda - \frac{mr}{2} - \frac{1}{2}\left(p + \frac{2}{3}\right)\right\}V_2 + \epsilon\sigma\eta(V_2)\zeta = 0, \tag{51}$$

for all V_2 on $M^3(\epsilon)$, where D stands for the gradient operator of g.

The covariant derivative (51) with respect to V_1 leads to

$$\begin{aligned} \nabla_{V_1}\nabla_{V_2}Df = &-l\left\{\left(\nabla_{V_1}Q\right)V_2 + Q\left(\nabla_{V_1}V_2\right)\right\} \\ &- \left\{\Lambda - \frac{mr}{2} - \frac{1}{2}\left(p + \frac{2}{3}\right)\right\}\nabla_{V_1}V_2 + m\frac{V_1(r)}{2}V_2 \\ &- \epsilon\sigma\left\{g(\varphi V_1, V_2)\zeta + \eta\left(\nabla_{V_1}V_2\right)\zeta + \epsilon\eta(V_2)\varphi V_1\right\}. \end{aligned} \tag{52}$$

Interchanging the role of V_1 and V_2 in (52), we have

$$\begin{aligned} \nabla_{V_2}\nabla_{V_1}Df = &-l\left\{\left(\nabla_{V_2}Q\right)V_1 + Q\left(\nabla_{V_2}V_1\right)\right\} \\ &- \left\{\Lambda - \frac{mr}{2} - \frac{1}{2}\left(p + \frac{2}{3}\right)\right\}\nabla_{V_2}V_1 + m\frac{V_2(r)}{2}V_1 \\ &- \epsilon\sigma\left\{g(\varphi V_2, V_1)\zeta + \eta\left(\nabla_{V_2}V_1\right)\zeta + \epsilon\eta(V_1)\varphi V_2\right\}. \end{aligned} \tag{53}$$

By using (51)–(53), the well-known relation $R(V_1, V_2)Df = \nabla_{V_1}\nabla_{V_2}Df - \nabla_{V_2}\nabla_{V_1}Df - \nabla_{[V_1,V_2]}Df$ takes the form

$$\begin{aligned} R(V_1, V_2)Df = &\, l\left\{\left(\nabla_{V_2}Q\right)V_1 - \left(\nabla_{V_1}Q\right)V_2\right\} \\ &+ \frac{m}{2}\{V_1(r)V_2 - V_2(r)V_1\} \\ &+ \sigma\{\eta(V_1)\varphi V_2 - \eta(V_2)\varphi V_1\}. \end{aligned} \tag{54}$$

The covariant differentiation of (22) with respect to V_2 gives

$$\begin{aligned} \left(\nabla_{V_2}Q\right)V_1 = &\frac{V_2(r)}{2}(V_1 + \eta(V_1)\zeta) \\ &+ \left(\frac{r}{2} - 3\epsilon\right)\left(g(\varphi V_1, V_2)\zeta + \epsilon\eta(V_1)\varphi V_2\right), \end{aligned} \tag{55}$$

which by replacing $V_1 = \zeta$ then using (8) and (9) reduces to

$$\left(\nabla_{V_2}Q\right)\zeta = -\left(\frac{\epsilon r}{2} - 3\right)\varphi V_2. \tag{56}$$

Again, replacing V_2 by ζ in (55) and using (9), we find

$$\left(\nabla_\zeta Q\right)V_1 = \frac{(\zeta r)}{2}(V_1 + \eta(V_1)\zeta). \tag{57}$$

Subtracting (57) from (56), we find

$$\left(\nabla_{V_2}Q\right)\zeta - \left(\nabla_\zeta Q\right)V_1 = -\left(\frac{\epsilon r}{2} - 3\right)\varphi V_2 - \frac{(\zeta r)}{2}(V_1 + \eta(V_1)\zeta). \tag{58}$$

Now, putting $V_1 = \zeta$ in (54) then using (8) and (9), we have

$$R(\zeta, V_2)Df = l\left\{\left(\nabla_{V_2}Q\right)\zeta - \left(\nabla_\zeta Q\right)V_2\right\} + \frac{m}{2}\{\zeta(r)V_2 - V_2(r)\zeta\} - \sigma\varphi V_2. \tag{59}$$

Taking the inner product of foregoing equation with ζ and using (58), we infer

$$g(R(\zeta, V_2)Df, \zeta) = \frac{\epsilon m}{2}\{\zeta(r)\eta(V_2) + V_2(r)\}. \qquad (60)$$

From relation (15), we have

$$g(R(\zeta, V_2)Df, \zeta) = -(V_2 f) - \zeta(f)\eta(V_2). \qquad (61)$$

By combining equations (60) and (61), it follows that $(V_2 f) + \{\zeta f + (\epsilon m\zeta(r)/2)\}\eta(V_2) + (\epsilon m/2)V_2(r) = 0$ for any V_2 on $M^3(\epsilon)$. Therefore, for r constant, we have

$$U = Df = -\epsilon(\zeta f)\zeta. \qquad (62)$$

This informs that the vector field U is pointwise collinear with ζ.

Now, taking the covariant derivative of (62) with respect to V_1, we have

$$\nabla_{V_1} Df = -\epsilon\{V_1(\zeta f)\zeta\} - (\zeta f)\varphi V_1. \qquad (63)$$

The inner product of (63) with ζ gives

$$g(\nabla_{V_1} Df, \zeta) = V_1(\zeta f). \qquad (64)$$

From (63) and (64), we arrive

$$\nabla_{V_1} Df = -\epsilon g(\nabla_{V_1} Df, \zeta)\zeta - (\zeta f)\varphi V_1. \qquad (65)$$

The inner product of (51) with ζ leads to $g(\nabla_{V_1} Df, \zeta) = \{-2l - \epsilon\Lambda + \sigma + (\epsilon mr/2) + (\epsilon/2)(p + (2/3))\}\eta(V_1)$, which in view of (40) reduces to

$$g(\nabla_{V_1} Df, \zeta) = 0. \qquad (66)$$

Thus, (51) together with (65) and (66) takes the form

$$QV_1 = -\frac{1}{l}\left\{\Lambda - \frac{mr}{2} - \frac{1}{2}\left(p + \frac{2}{3}\right)\right\}V_1 - \frac{\epsilon\sigma}{l}\eta(V_1)\zeta + \frac{1}{l}(\zeta f)\varphi V_1, \quad l \neq 0. \qquad (67)$$

This informs that $M^3(\epsilon)$ is a generalized conformal η-Einstein manifold.

Next, from (51) and (63), we have

$$lQV_1 + \left\{\Lambda - \frac{mr}{2} - \frac{1}{2}\left(p + \frac{2}{3}\right)\right\}V_1 + \epsilon\sigma\eta(V_1)\zeta = \epsilon\{V_1(\zeta f)\zeta\} + \epsilon(\zeta f)\varphi V_1. \qquad (68)$$

By putting $V_1 = \zeta$ in (68) then using (8), (9), and (17), we find

$$\left\{2\epsilon l + \Lambda - \epsilon\sigma - \frac{mr}{2} - \frac{1}{2}\left(p + \frac{2}{3}\right)\right\}\zeta = \epsilon\{\zeta(\zeta f)\zeta\}. \qquad (69)$$

The inner product of (69) with ζ and the use of (9) and (40) leads to $\zeta(\zeta f) = 0$.

If possible, we suppose that $\zeta = \partial/\partial t$ then the above equation takes the form

$$\frac{\partial^2 f}{\partial t^2} = 0. \qquad (70)$$

It is noticed that the potential function $f = d_1 + td_2$ where d_1 and d_2 are independent of t, satisfies equation (70). By considering the above facts, we can state the following.

Theorem 11. *Let an $M^3(\epsilon)$ admit a gradient CERYS. Then,*

(i) $M^3(\epsilon)$ is a generalized conformal η-Einstein manifold

(ii) the gradient of the potential function f is pointwise collinear with the Reeb vector field ζ and f satisfies equation (70) and it is governed by $f = d_1 + td_2$.

Example 1. We consider the manifold $M^3 = \{(u_1, u_2, u_3) \in R^3\}$, where (u_1, u_2, u_3) are the usual coordinates in R^3. Let κ_1, κ_2, and κ_3 be the vector fields on M^3 given by

$$\begin{aligned}\kappa_1 &= \cosh u_3 \frac{\partial}{\partial u_1} + \sinh u_3 \frac{\partial}{\partial u_2}, \kappa_2 \\ &= \sinh u_3 \frac{\partial}{\partial u_1} + \cosh u_3 \frac{\partial}{\partial u_2}, \kappa_3 = \epsilon\frac{\partial}{\partial u_3} = \zeta,\end{aligned} \qquad (71)$$

and these are linearly independent at each point of M^3. Let g be the Lorentzian metric defined by

$$g(\kappa_i, \kappa_j) = \begin{cases} 1, & \text{for } 1 \le i \le 2, \\ -\epsilon, & \text{for } i = j = 3, \\ 0, & \text{otherwise.} \end{cases} \qquad (72)$$

We define η, a 1-form as $\eta(V_1) = \epsilon g(V_1, \kappa_3)$ for all V_1 on M^3. Let φ be the $(1, 1)$ tensor field defined by

$$\varphi\kappa_1 = -\kappa_2, \quad \varphi\kappa_2 = -\kappa_1, \quad \varphi\kappa_3 = 0. \qquad (73)$$

Using the linearity of φ and g, we yield

$$\begin{aligned}\eta(\kappa_3) &= -1, \varphi^2 V_1 = V_1 + \eta(V_1)\zeta, g(\varphi V_1, \varphi V_2) \\ &= g(V_1, V_2) - \epsilon\eta(V_1)\eta(V_2),\end{aligned} \qquad (74)$$

for all V_1, V_2 on M^3

Now, by direct computations, we obtain

$$[\kappa_1, \kappa_2] = 0, \quad [\kappa_2, \kappa_3] = -\epsilon\kappa_1, \quad [\kappa_1, \kappa_3] = -\epsilon\kappa_2. \qquad (75)$$

By using well-known Koszul's formula, we find

$$\nabla_{\kappa_1}\kappa_1 = 0, \quad \nabla_{\kappa_2}\kappa_1 = -\kappa_3, \quad \nabla_{\kappa_3}\kappa_1 = 0, \quad \nabla_{\kappa_1}\kappa_2 = -\kappa_3, \quad \nabla_{\kappa_2}\kappa_2 = 0,$$
$$\nabla_{\kappa_3}\kappa_2 = 0, \quad \nabla_{\kappa_1}\kappa_3 = -\epsilon\kappa_2, \quad \nabla_{\kappa_2}\kappa_3 = -\epsilon\kappa_1, \quad \nabla_{\kappa_3}\kappa_3 = 0. \tag{76}$$

Let $V_1 = V_1{}^1\kappa_1 + V_1{}^2\kappa_2 + V_1{}^3\kappa_3$ and $V_2 = V_2{}^1\kappa_1 + V_2{}^2\kappa_2 + V_2{}^3\kappa_3$ be the vector fields on M^3. Then, for $\kappa_3 = \zeta$ one can easily verify that

$$\nabla_{V_1}\zeta = \epsilon\varphi V_1 \quad \text{and} \quad (\nabla_{V_1}\varphi)V_2 = g(V_1, V_2)\zeta + \epsilon\eta(V_2)V_1 + 2\epsilon\eta(V_1)\eta(V_2)\zeta. \tag{77}$$

Thus, the manifold M^3 is an ϵ-LP-Sasakian 3-manifold.

By using the above results, we can easily obtain the following components of the curvature tensor R:

$$R(\kappa_1, \kappa_2)\kappa_1 = \epsilon\kappa_2, \quad R(\kappa_1, \kappa_2)\kappa_2 = -\epsilon\kappa_1, \quad R(\kappa_1, \kappa_2)\kappa_3 = 0,$$
$$R(\kappa_2, \kappa_3)\kappa_1 = 0, \quad R(\kappa_2, \kappa_3)\kappa_2 = -\epsilon\kappa_3, \quad R(\kappa_2, \kappa_3)\kappa_3 = -\kappa_2,$$
$$R(\kappa_1, \kappa_3)\kappa_1 = -\epsilon\kappa_3, \quad R(\kappa_1, \kappa_3)\kappa_2 = 0, \quad R(\kappa_1, \kappa_3)\kappa_3 = -\kappa_1. \tag{78}$$

We calculate the Ricci tensors as follows:

$$S(\kappa_1, \kappa_1) = S(\kappa_2, \kappa_2) = 0, \quad S(\kappa_3, \kappa_3) = -2 \Longrightarrow r = 2. \tag{79}$$

By putting $V_1 = V_2 = \kappa_3$ in (42) and using $S(\kappa_3, \kappa_3) = -2$, it follows that

$$\Lambda - \epsilon\sigma = -2\epsilon l + \frac{mr}{2} + \frac{1}{2}\left(p + \frac{2}{3}\right). \tag{80}$$

Again putting $V_1 = V_2 = \kappa_1$ in (42) and using $S(\kappa_1, \kappa_1) = 0$, we obtain $\Lambda = (mr/2) + (1/2)(p + (2/3))$. Thus, from (80), we find $\sigma = 2l$. Hence, we can say that for $\Lambda = (mr/2) + (1/2)(p + (2/3))$ and $\sigma = 2l$, the data $(g, \zeta, l, m, \Lambda, \sigma)$ defines a CERYS on the manifold $(M^3, \varphi, \zeta, \eta, g, \epsilon)$.

Data Availability

No data is used in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] A. Bejancu and K. L. Duggal, "Real hypersurfaces of indefinite Kaehler manifolds," *International Journal of Mathematics and Mathematical Sciences*, vol. 16, no. 3, p. 556, 1993.

[2] X. Xufeng and C. Xiaoli, "Two theorems on ϵ-Sasakian manifolds," *International Journal of Mathematics and Mathematical Sciences*, vol. 21, no. 2, p. 254, 1998.

[3] U. C. De and A. Sarkar, "On ϵ-Kenmotsu manifold," *Hardonic Journal*, vol. 32, pp. 231–242, 2009.

[4] A. Haseeb, A. Prakash, and M. D. Siddiqi, "On a quarter-symmetric metric connection in an ϵ-Lorentzian para-Sasakian manifold," *Acta Mathematica Universitatis Comenianae*, vol. 86, pp. 143–152, 2017.

[5] D. H. Jin, "Geometry of lightlike hypersurfaces of an indefinite Sasakian manifold," *Indian Journal of Pure and Applied Mathematics*, vol. 41, no. 4, pp. 569–581, 2010.

[6] R. Prasad and V. Srivastava, "On ϵ-Lorentzian para-Sasakian manifolds," *Korean Mathematical Society*, vol. 27, no. 2, pp. 297–306, 2012.

[7] C. Yildirim and B. Sahin, "Transversal lightlike submanifolds of indefinite Sasakian manifolds," *Turkish Journal of Mathematics*, vol. 34, pp. 561–583, 2010.

[8] A. E. Fischer, "An introduction to conformal Ricci flow," *Classical and Quantum Gravity*, vol. 21, no. 3, pp. 171–218, 2004.

[9] N. Basu and A. Bhattacharyya, "Conformal Ricci soliton in Kenmotsu manifold," *Global Journal of Advanced Research on Classical and Modern Geometries*, vol. 4, pp. 15–21, 2015.

[10] S. Guler and M. Crasmareanu, "Ricci-Yamabe maps for Riemannian flows and their volume variation and volume entropy," *Turkish Journal of Mathematics*, vol. 43, no. 5, pp. 2631–2641, 2019.

[11] U. C. De, A. Sardar, and K. De, "Ricci-Yamabe solitons and 3-dimensional Riemannian manifolds," *Turkish Journal of Mathematics*, vol. 46, no. 3, pp. 1078–1088, 2022.

[12] P. Zhang, Y. Li, S. Roy, S. Dey, and A. Bhattacharyya, "Geometrical structure in a perfect fluid spacetime with conformal Ricci-Yamabe soliton," *Symmetry*, vol. 14, no. 3, p. 594, 2022.

[13] A. M. Blaga, S. Y. Perktas, B. E. Acet, and F. E. Erdogan, "η-Ricci solitons in (ε)-almost paracontact metric manifolds," *Matematicki Glasnik*, vol. 53, no. 1, pp. 205–220, 2018.

[14] M. Brozos-Vazquez, G. Calvaruso, E. Garcia-Rio, and S. Gavino-Fernandez, "Three-dimensional Lorentzian homogeneous Ricci solitons," *Israel Journal of Mathematics*, vol. 188, no. 1, pp. 385–403, 2012.

[15] G. Catino and L. Mazzieri, "Gradient Einstein solitons," *Nonlinear Analysis*, vol. 132, pp. 66–94, 2016.

[16] J. T. Cho and M. Kimura, "Ricci solitons and real hypersurfaces in a complex space form," *Tohoku Mathematical Journal, Second Series*, vol. 61, no. 2, pp. 205–212, 2009.

[17] U. C. De and A. Haseeb, "Conformal η-Ricci solitons on ϵ-para Sasakian manifolds," *Journal: Novi Sad Journal of Mathematics*, vol. 21, 2021.

[18] R. S. Hamilton, *Lectures on Geometric Flows Unpublished Manuscript*, 1989.

[19] R. S. Hamilton, "The Ricci flow on surfaces, Mathematics and general relativity,," *Contemporary Mathematics*, vol. 71, pp. 237–262, 1988.

[20] A. Haseeb and M. A. Akyol, "On ϵ-Kenmotsu 3-manifolds admitting $*$-conformal η-Ricci solitons," *Balkan Journal of Geometry and its Applications*, vol. 26, no. 1, pp. 1–11, 2021.

[21] A. Haseeb and U. C. De, "η-Ricci solitons in ϵ-Kenmotsu manifolds," *Journal of Geometry*, vol. 110, no. 2, pp. 1–12, 2019.

[22] A. Haseeb and R. Prasad, "η-Ricci solitons in ϵ-LP-Sasakian manifolds with a quarter symmetric metric connection," *Honam Mathematical Journal*, vol. 41, pp. 539–558, 2019.

[23] A. Haseeb and R. Prasad, "Conformal η-Ricci solitons in ϵ-Kenmotsu manifolds," *Publications of the Mathematical Institute, New series*, vol. 108, no. 122, pp. 91–102, 2020.

[24] Y. Li, D. Ganguly, S. Dey, and A. Bhattacharyya, "Conformal η-Ricci solitons within the framework of indefinite Kenmotsu manifolds," *AIMS Mathematics*, vol. 7, no. 4, pp. 5408–5430, 2022.

[25] H. I. Yoldas, "On Kenmotsu manifolds admitting η-Ricci-Yamabe solitons," *International Journal of Geometric Methods in Modern Physics*, vol. 18, no. 12, p. 2150189, 2021.

[26] K. Yano, *Integral Formulas in Riemannian Geometry*, Marcel Dekker, Inc., New York, 1970.

[27] K. Yano, "On the torse-forming directions in Riemannian spaces," *Proceedings of the Imperial Academy*, vol. 20, no. 6, pp. 340–345, 1944.