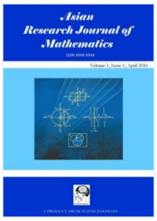

Asian Research Journal of Mathematics

5(1): 1-9, 2017; Article no.ARJOM.32884

ISSN: 2456-477X

Some Power Sums from the Geometric Series

Aeran Kim[1*]

[1] *A Private Mathematics Academy, 23, Maebong 5-gil, Deokjin-gu, Jeonju-si, Jeollabuk-do, 54921, South Korea.*

Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/ARJOM/2017/32884

Editor(s):

(1) Rakesh Prakash Tripathi, Department of Mathematics, Graphic Era University, India.

(2) Subramanian Vaithyasubramanian, Department of Mathematics, Sathyabama University, India.

Reviewers:

(1) Clement Boateng Ampadu, Massachusetts, USA.

(2) Jitender Singh, Guru Nanak Dev University, India.

(3) Erdal Unluyol, Ordu University, Turkey.

(4) Ricardo Enguia, Instituto Superior de Engenharia de Lisboa, Portugal.

(5) Ali Yakar, Gaziosmanpasa University, Turkey.

Complete Peer review History: http://www.sciencedomain.org/review-history/19741

Received: ***21st March 2017***

Accepted: ***16th June 2017***

Published: ***28th June 2017***

Original Research Article

Abstract

We focus on the summation of $\sum_{r=1}^{n} \frac{(r+k-1)!}{(r-1)!} x^{r+k}$ and express it as simple polynomials and find a relation between them.

Keywords: Sequences; polynomials.

2010 Mathematics Subject Classification: 11B50,11C08.

**Corresponding author: E-mail: ae_ran_kim@hotmail.com;*

1 Introduction

Let $\mathbb{N}$ denote the set of positive integers. For $n, k \in \mathbb{N} \cup \{0\}$, the sum of powers of consecutive integers,

$$\sum_{r=0}^{n} r^k$$

was studied by Faulhaber, Fermat, Pascal, Bernoulli, Jacobi, and many other mathematicians. Recently Sullivan [1], Edwards [2], Scott [3], and Khan [4] have contributed on power sums. Moreover Gauthier [5] studied sums of the type

$$\sum_{r=0}^{n} r^k x^r,$$

where $n, k \geq 0$ are integers and x is an arbitrary parameter (real or complex). Gauthier obtained some results for the sums of powers of consecutive integers as a special case.

In this paper we focus on the following power sum

$$\sum_{r=1}^{n} \frac{(r+k-1)!}{(r-1)!} x^{r+k}. \tag{1.1}$$

After defining the differential operator $\mathcal{D} = x^2 \dfrac{d}{dx}$, we obtain some formulae for the summation (1.1), following Gauthier's method on $\sum r^k x^r$. More precisely, we deduce

Theorem 1.1. *Let $n, k \in \mathbb{N}$. Then*

$$\sum_{r=1}^{n} \frac{(r+k-1)!}{(r-1)!} x^{r+k} = x^{n+k} P_k(x; n) - x^k \cdot a_0^{(k)}(x),$$

where

$$\begin{aligned} P_k(x; n) &= \sum_{r=1}^{k+1} a_{r-1}^{(k)}(x) n^{r-1} \\ &= -\prod_{r=1}^{k} (n+r)x - \prod_{r=1}^{k} (n+r+1)x^2 - \cdots \end{aligned}$$

and $P_k(x; n)$ is a polynomial of degree k in n, with coefficients $a_{r-1}^{(k)}$ which depend on x.

Theorem 1.2. *Let $n, k \in \mathbb{N}$. Then*

$$x P_{k+1}(x; n) = (n+k) x P_k(x; n) + \mathcal{D} P_k(x; n)$$

and

$$x a_0^{(k+1)}(x) = k x a_0^{(k)}(x) + \mathcal{D} a_0^{(k)}(x).$$

2 Proofs of Theorem 1.1 and Theorem 1.2

Let $x \neq 1$ be an arbitrary real or complex parameter, and note the following identity,

$$\sum_{r=0}^{n} x^r = \frac{1-x^{n+1}}{1-x}. \tag{2.1}$$

By k successive applications of the differential operator $\mathcal{D} = x^2 \frac{d}{dx}$ to both sides of (2.1), we obtain as follows.

Lemma 2.1. *Let $n \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$. Then*

$$\sum_{r=1}^{n} \frac{(r+k-1)!}{(r-1)!} x^{r+k} = \begin{cases} \mathcal{D}^k \left(\frac{1-x^{n+1}}{1-x} - 1 \right), & \text{for } k = 0, \\ \mathcal{D}^k \left(\frac{1-x^{n+1}}{1-x} \right), & \text{for } k \geq 1. \end{cases}$$

Proof. For $k = 0$ the summation becomes Eq. (2.1) so it is right. For $k = 1$ we take $\mathcal{D} = x^2 \frac{d}{dx}$ and then

$$\mathcal{D}\left(\frac{1-x^{n+1}}{1-x}\right) = x^2 \frac{d}{dx}\left(\sum_{r=0}^{n} x^r\right) = \sum_{r=0}^{n} r x^{r+1} = \sum_{r=1}^{n} r x^{r+1}.$$

We suppose that

$$\sum_{r=1}^{n} \frac{(r+k-1)!}{(r-1)!} x^{r+k} = \mathcal{D}^k \left(\frac{1-x^{n+1}}{1-x}\right).$$

Then

$$\begin{aligned} \mathcal{D}^{k+1}\left(\frac{1-x^{n+1}}{1-x}\right) = \mathcal{D}\left(\mathcal{D}^k\left(\frac{1-x^{n+1}}{1-x}\right)\right) &= \mathcal{D}\left(\sum_{r=1}^{n} \frac{(r+k-1)!}{(r-1)!} x^{r+k}\right) \\ &= \sum_{r=1}^{n} \frac{(r+k-1)!}{(r-1)!} \cdot (r+k) x^{r+k+1} \\ &= \sum_{r=1}^{n} \frac{(r+k)!}{(r-1)!} x^{r+k+1}. \end{aligned}$$

□

Example 2.2. *Let $k = 1$ in Lemma 2.1.*

$$\sum_{r=1}^{n} r x^{r+1} = \sum_{r=1}^{n} \frac{r!}{(r-1)!} x^{r+1} = \mathcal{D}\left(\frac{1-x^{n+1}}{1-x}\right) = \frac{n x^{n+3} - (n+1)x^{n+2} + x^2}{(1-x)^2}$$

and so if $x = 2$ then we have

$$\sum_{r=1}^{n} r \cdot 2^{r+1} = 2^{n+3}n - 2^{n+2}(n+1) + 2^2$$

and if $x = 3$ then we obtain

$$\sum_{r=1}^{n} r \cdot 3^{r+1} = \frac{3^{n+3}n - 3^{n+2}(n+1) + 3^2}{2^2}.$$

In a similar manner, after putting $k = 2$ in Lemma 2.1, we substitute $x = 2$ and $x = 3$, respectively then we have

$$\begin{aligned}&\sum_{r=1}^{n} r(r+1) \cdot 2^{r+2} \\ &= 2^3 \left\{-2^{n+3}n + 2^{n+2}(n+1) + 2^{n+1}n(n+3) - 2^n(n+1)(n+2) - 2\right\}\end{aligned}$$

and

$$\begin{aligned}&\sum_{r=1}^{n} r(r+1) \cdot 3^{r+2} \\ &= \frac{3^3}{2^2}\left\{-3^{n+2}n + 3^{n+1}(n^2+4n+1) - 3^n(n+1)(n+2) - 1\right\}.\end{aligned}$$

Proof of Theorem 1.1. We can rewrite Lemma 2.1 as

$$\begin{aligned}\sum_{r=1}^{n} \frac{(r+k-1)!}{(r-1)!} x^{r+k} &= \mathcal{D}^k\left(\frac{1-x^{n+1}}{1-x}\right) \\ &= \mathcal{D}^k\left(\frac{x^{n+1}}{x-1}\right) - \mathcal{D}^k\left(\frac{1}{x-1}\right) \\ &= \mathcal{D}^k\left(x^{n+1}(-1-x-x^2-x^3-\cdots)\right) \\ &\quad - \mathcal{D}^k\left(-1-x-x^2-x^3-\cdots\right) \\ &= \mathcal{D}^k\left(-x^{n+1}-x^{n+2}-x^{n+3}-\cdots\right) \\ &\quad - \mathcal{D}^k\left(-1-x-x^2-x^3-\cdots\right).\end{aligned} \tag{2.2}$$

Then, since

$$\begin{aligned}\mathcal{D}\left(-x^{n+1}-x^{n+2}-x^{n+3}-\cdots\right) &= -(n+1)x^{n+2} - (n+2)x^{n+3} - \cdots, \\ \mathcal{D}^2\left(-x^{n+1}-x^{n+2}-x^{n+3}-\cdots\right) &= -(n+1)(n+2)x^{n+3} \\ &\quad - (n+2)(n+3)x^{n+4} - \cdots, \\ &\vdots \\ \mathcal{D}^k\left(-x^{n+1}-x^{n+2}-x^{n+3}-\cdots\right) &= -\prod_{r=1}^{k}(n+r)x^{n+k+1} - \prod_{r=1}^{k}(n+r+1)x^{n+k+2} - \cdots\end{aligned}$$

and

$$\mathcal{D}\left(-1-x-x^2-x^3-\cdots\right)=-x^2-2x^3-3x^4-\cdots,$$
$$\mathcal{D}^2\left(-1-x-x^2-x^3-\cdots\right)=-1\cdot 2x^3-2\cdot 3x^4-3\cdot 4x^5-\cdots,$$
$$\vdots$$
$$\mathcal{D}^k\left(-1-x-x^2-x^3-\cdots\right)=-\prod_{r=1}^{k} rx^{k+1}-\prod_{r=1}^{k}(r+1)x^{k+2}-\cdots,$$

the Eq. (2.2) becomes

$$\begin{aligned}
&\sum_{r=1}^{n}\frac{(r+k-1)!}{(r-1)!}x^{r+k}\\
&=\left\{-\prod_{r=1}^{k}(n+r)x^{n+k+1}-\prod_{r=1}^{k}(n+r+1)x^{n+k+2}-\cdots\right\}\\
&\quad-\left\{-\prod_{r=1}^{k} rx^{k+1}-\prod_{r=1}^{k}(r+1)x^{k+2}-\cdots\right\}\\
&=x^{n+k}\left\{-\prod_{r=1}^{k}(n+r)x-\prod_{r=1}^{k}(n+r+1)x^2-\cdots\right\}\\
&\quad-x^k\left\{-\prod_{r=1}^{k} rx-\prod_{r=1}^{k}(r+1)x^2-\cdots\right\}\\
&=x^{n+k}\sum_{r=1}^{k+1}a_{r-1}^{(k)}(x)n^{r-1}-x^k\cdot a_0^{(k)}(x)\\
&=x^{n+k}P_k(x;n)-x^k\cdot a_0^{(k)}(x).
\end{aligned}\tag{2.3}$$

□

Corollary 2.3.

$$\sum_{r=1}^{n}\frac{(r+k)!}{(r-1)!}x^{r+k+1}=\mathcal{D}\sum_{r=1}^{n}\frac{(r+k-1)!}{(r-1)!}x^{r+k}.$$

Proof. We note that

$$\begin{aligned}
\mathcal{D}\sum_{r=1}^{n}\frac{(r+k-1)!}{(r-1)!}x^{r+k}&=x^2\frac{d}{dx}\sum_{r=1}^{n}\frac{(r+k-1)!}{(r-1)!}x^{r+k}\\
&=x^2\sum_{r=1}^{n}\frac{(r+k-1)!}{(r-1)!}(r+k)x^{r+k-1}\\
&=\sum_{r=1}^{n}\frac{(r+k)!}{(r-1)!}x^{r+k+1},
\end{aligned}$$

which completes the proof.

□

Proof of Theorem 1.2. Using Theorem 1.1 and Corollary 2.3, we can easily know that

$$\begin{aligned}
&x^{n+k+1}P_{k+1}(x;n) - x^{k+1}a_0^{(k+1)}(x) \\
&= \sum_{r=1}^{n} \frac{(r+k)!}{(r-1)!} x^{r+k+1} \\
&= \mathcal{D} \sum_{r=1}^{n} \frac{(r+k-1)!}{(r-1)!} x^{r+k} \\
&= \mathcal{D}\left(x^{n+k}P_k(x;n) - x^k a_0^{(k)}(x)\right) \\
&= \left(\mathcal{D}x^{n+k}\right) P_k(x;n) + x^{n+k}\mathcal{D}P_k(x;n) - \left(\mathcal{D}x^k\right) a_0^{(k)}(x) - x^k \mathcal{D}a_0^{(k)}(x) \\
&= (n+k)x^{n+k+1}P_k(x;n) + x^{n+k}\mathcal{D}P_k(x;n) - kx^{k+1}a_0^{(k)}(x) - x^k\mathcal{D}a_0^{(k)}(x)
\end{aligned}$$

and so

$$\begin{aligned}
&x^{n+1}P_{k+1}(x;n) - xa_0^{(k+1)}(x) \\
&= (n+k)x^{n+1}P_k(x;n) + x^n\mathcal{D}P_k(x;n) - kxa_0^{(k)}(x) - \mathcal{D}a_0^{(k)}(x).
\end{aligned}$$

This leads that

$$\begin{aligned}
&x^n\left(xP_{k+1}(x;n) - (n+k)xP_k(x;n) - \mathcal{D}P_k(x;n)\right) \\
&= xa_0^{(k+1)}(x) - kxa_0^{(k)}(x) - \mathcal{D}a_0^{(k)}(x).
\end{aligned}$$

The right hand side of the above identity is independent of n but the left hand side has a factor which grows exponentially with n. Consequently, for the identity to hold for all values of n, with x fixed but arbitrary, we must have

$$xP_{k+1}(x;n) - (n+k)xP_k(x;n) - \mathcal{D}P_k(x;n) = 0$$

and

$$xa_0^{(k+1)}(x) - kxa_0^{(k)}(x) - \mathcal{D}a_0^{(k)}(x) = 0.$$

Therefore we conclude that

$$xP_{k+1}(x;n) = (n+k)xP_k(x;n) + \mathcal{D}P_k(x;n)$$

and

$$xa_0^{(k+1)}(x) = kxa_0^{(k)}(x) + \mathcal{D}a_0^{(k)}(x).$$

□

Example 2.4. *Consider the following equation deduced from Theorem 1.2 :*

$$xP_2(x;n) = (n+1)xP_1(x;n) + \mathcal{D}P_1(x;n). \tag{2.4}$$

Then by Eq. (2.3), the left hand side of (2.4) is

$$\begin{aligned} xP_2(x;n) &= x\left\{-\prod_{r=1}^{2}(n+r)x-\prod_{r=1}^{2}(n+r+1)x^2-\cdots\right\} \\ &= x\left\{-(n+1)(n+2)x-(n+2)(n+3)x^2-\cdots\right\} \\ &= -(n+1)(n+2)x^2-(n+2)(n+3)x^3-\cdots \end{aligned}$$

and the right hand side of (2.4) is

$$\begin{aligned} &(n+1)xP_1(x;n)+\mathcal{D}P_1(x;n) \\ &= (n+1)x\left\{-\prod_{r=1}^{1}(n+r)x-\prod_{r=1}^{1}(n+r+1)x^2-\cdots\right\} \\ &\quad +\mathcal{D}\left\{-\prod_{r=1}^{1}(n+r)x-\prod_{r=1}^{1}(n+r+1)x^2-\cdots\right\} \\ &= (n+1)x\left\{-(n+1)x-(n+2)x^2-\cdots\right\}+\mathcal{D}\left\{-(n+1)x-(n+2)x^2-\cdots\right\} \\ &= -(n+1)^2x^2-(n+1)(n+2)x^3-\cdots+x^2\left\{-(n+1)-2(n+2)x-\cdots\right\} \\ &= -(n+1)(n+2)x^2-(n+2)(n+3)x^3-\cdots \end{aligned}$$

therefore it is shown to be right. Similarly we have

$$xP_3(x;n)=(n+2)xP_2(x;n)+\mathcal{D}P_2(x;n). \tag{2.5}$$

Then the left hand side of (2.5) is

$$\begin{aligned} xP_3(x;n) &= x\left\{-\prod_{r=1}^{3}(n+r)x-\prod_{r=1}^{3}(n+r+1)x^2-\cdots\right\} \\ &= x\left\{-(n+1)(n+2)(n+3)x-(n+2)(n+3)(n+4)x^2-\cdots\right\} \\ &= -(n+1)(n+2)(n+3)x^2-(n+2)(n+3)(n+4)x^3-\cdots \end{aligned}$$

and the right hand side of (2.5) is

$$\begin{aligned} &(n+2)xP_2(x;n)+\mathcal{D}P_2(x;n) \\ &= (n+2)x\left\{-\prod_{r=1}^{2}(n+r)x-\prod_{r=1}^{2}(n+r+1)x^2-\cdots\right\} \\ &\quad +\mathcal{D}\left\{-\prod_{r=1}^{2}(n+r)x-\prod_{r=1}^{2}(n+r+1)x^2-\cdots\right\} \\ &= (n+2)x\left\{-(n+1)(n+2)x-(n+2)(n+3)x^2-\cdots\right\} \\ &\quad +\mathcal{D}\left\{-(n+1)(n+2)x-(n+2)(n+3)x^2-\cdots\right\} \\ &= -(n+1)(n+2)^2x^2-(n+2)^2(n+3)x^3-\cdots \\ &\quad +x^2\left\{-(n+1)(n+2)-2(n+2)(n+3)x-\cdots\right\} \\ &= -(n+1)(n+2)(n+3)x^2-(n+2)(n+3)(n+4)x^3-\cdots. \end{aligned}$$

Lemma 2.5. *Let $n, k \in \mathbb{N}$. Then*

$$xa_{r-1}^{(k+1)}(x) = xa_{r-2}^{(k)}(x) + xka_{r-1}^{(k)}(x) + \mathcal{D}a_{r-1}^{(k)}(x).$$

Proof. In advance we define

$$a_{k+1}^{(k)} := 0 \qquad \text{and} \qquad a_{-1}^{(k)} := 0. \tag{2.6}$$

Now by Theorem 1.1, Theorem 1.2, and (2.6) we have

$$\begin{aligned}
x\sum_{r=1}^{k+2} a_{r-1}^{(k+1)}(x)n^{r-1} &= xP_{k+1}(x;n) \\
&= (n+k)xP_k(x;n) + \mathcal{D}P_k(x;n) \\
&= (n+k)x\sum_{r=1}^{k+1} a_{r-1}^{(k)}(x)n^{r-1} + \mathcal{D}\sum_{r=1}^{k+1} a_{r-1}^{(k)}(x)n^{r-1} \\
&= x\left\{\sum_{r=1}^{k+1} a_{r-1}^{(k)}(x)n^{r} + \sum_{r=1}^{k+1} ka_{r-1}^{(k)}(x)n^{r-1}\right\} + \mathcal{D}\sum_{r=1}^{k+1} a_{r-1}^{(k)}(x)n^{r-1} \\
&= x\left\{\sum_{R=2}^{k+2} a_{R-2}^{(k)}(x)n^{R-1} + \sum_{r=1}^{k+1} ka_{r-1}^{(k)}(x)n^{r-1}\right\} + \mathcal{D}\sum_{r=1}^{k+1} a_{r-1}^{(k)}(x)n^{r-1} \\
&= x\left\{\sum_{R=1}^{k+2} a_{R-2}^{(k)}(x)n^{R-1} + \sum_{r=1}^{k+2} ka_{r-1}^{(k)}(x)n^{r-1}\right\} + \mathcal{D}\sum_{r=1}^{k+2} a_{r-1}^{(k)}(x)n^{r-1} \\
&= \sum_{r=1}^{k+2}\left\{xa_{r-2}^{(k)}(x) + xka_{r-1}^{(k)}(x) + \mathcal{D}a_{r-1}^{(k)}(x)\right\}n^{r-1}
\end{aligned}$$

and so

$$xa_{r-1}^{(k+1)}(x) = xa_{r-2}^{(k)}(x) + xka_{r-1}^{(k)}(x) + \mathcal{D}a_{r-1}^{(k)}(x).$$

□

Remark 2.1. If $r = 1$ in Lemma 2.5 then by (2.6) we obtain

$$\begin{aligned}
xa_0^{(k+1)}(x) &= xa_{-1}^{(k)}(x) + xka_0^{(k)}(x) + \mathcal{D}a_0^{(k)}(x) \\
&= xka_0^{(k)}(x) + \mathcal{D}a_0^{(k)}(x),
\end{aligned}$$

which confirms Theorem 1.2.

3 Conclusion

Note [6] for more information on power sums. We start this article from the geometric sum

$$\sum_{r=0}^{n} x^r = \frac{1-x^{n+1}}{1-x}$$

and consider the summation $\sum_{r=1}^{n} \frac{(r+k-1)!}{(r-1)!} x^{r+k}$ to express it as simple polynomials. Moreover as we can see, Lemma 2.1 enables us to calculate the complex summation easily.

Competing Interests

Author has declared that no competing interests exist.

References

[1] Sullivan D. The sums of powers of integers. Math. Gazette. 1987;71:144-146.

[2] Edwards AWF. A quick route to the sums of powers. Amer. Math. Monthly. 1986;93:451455.

[3] Scott SH. Sums of powers of natural numbers: (1) By coefficient operation. Math. Gazette. 1980;64:231-238.

[4] Khan RA. A simple derivation of a formula for $\sum_{k=1}^{n} k^r$. The Fibonacci Quarterly. 1981;19.2:177-180.

[5] Gauthier N. Derivation of a formula for $\sum r^k x^r$. The Fibonacci Quarterly. 1989;27(5):402-408.

[6] Singh J. Defining power sums of n and $\varphi(n)$ integers. International Journal of Number Theory. 2009;41-53.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://sciencedomain.org/review-history/19741