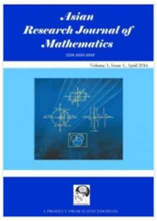

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Properties and Applications of Some Algebraic Transformations from the Conditional Function

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

The intended purpose of this paper is to associate with a given function $y = f(x)$ whose derivative admits one to three turning points an infinite number of other functions, called conditional functions (Cd_f),which are related to f in a way similar to that of a bijective function and its inverse. However, the new application called conditional function is defined for both bijective and non bijective functions. The composite map of f and its conditional function is called a transformation , and some applications of these transformations presented in the paper include amongst others, the determination of analytic solutions for a number of algebraic equations describing the dynamics of natural phenomena.

Keywords: Conditional function; New methods; transformations; natural phenomena.

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1 Introduction

Injective or one-to-one functions are those such that preimages of elements of the range are unique. In other words, every element in the range is assigned to exactly one element in the domain. However, a function is surjective or onto if the range is equal to the codomain. Both injective and surjective function is a bijective function [1]. Among these functions those whose derivative admits one to three turning points are numerous. Some of them are elementary functions such that logarithm, exponential, polynomials...[2]. The elementary functions of a real variable possess properties that could greatly simplify the mathematical analysis needed to be done on them. Also, many problems in mathematics deal with elementary functions or even if the functions are non-elementary, very often the studying of these non-elementary functions leads to elementary functions. It is also the case for special functions which are numerous with applications in many branches of sciences. One of the special functions is the series $w(x)$ converging for $|x| < 1/e$ where $w(x)$ is defined to be a function satisfying $W(x)e^{W(x)} = x$. In literature, the solutions of equation $xe^x = y$ are expressed by the function Omega (W) which has two branches [3]. It can also be expressed in terms of tree function T satisfying $T(x)e^{-T(x)} = x$ [4]. In addition, The *glog* function bears a strong resemblance to W, possessing similar properties and useful common applications as enumeration of trees, enzyme kinetics, linear delay equations, combustion, population growth, spread of disease, and the analysis of algorithms [3][5][6][7]. Several other cases involve generalized Gaussian noise, solar winds, black holes, general relativity, quantum chromodynamics, fuel consumption, Stirlings formula for $n!$, cardiorespiratory control, water-wave heights in oceanography, enumeration of trees in combinatorics, and statistical mechanics [8][9][10]. However, the field application of these elementary and special functions is limited. For example the Omega (W), Tree, *glog* and many other related, functions can not be used to describe the dynamics of certain natural phenomena such that solution of the equation governing the dilaton field, from which is derived the metric of the $R = T$ or lineal two-body gravity problem in $1 + 1$ dimensions (one spatial dimension and one time dimension)[12]. Another example among others is the equation $(ax + b)p^{cx+d} + (gx + h) = 0[(a, c) \in \Re^{2*}, (b, d, g, h) \in \Re^4, p > 0]$ whose these special functions can not be used to finding differents classes of analytic solutions except for the case $g = 0$. To overcome this problem, in this study, a given function $y = f(x)$ whose derivative admits one to three turning points was associated to an infinite number of other functions, called conditional functions (Cd_f),which are related to f in a way similar to that of a bijective function and its inverse. The new application has many advantages in the fact that all the above previous functions could be expressed in terms of conditional function. It becomes a unified model to define and characterize all the functions whose derivative admits zero to three turning points.

Definition 1.1. Let $y = g(x)$, and $y = f(x)$, be a given functions defined in their domain. Conditional function $h = g(x)$ is an infinite number of function associated to $y = f(x)$ whose derivative does not have a minimum or maximum or admits one to many turning points, which are related to f in a way similar to that of a bijective function and its inverse. It is also defined for both bijective and non bijective functions.

Notation 1.1. *The conditional function is denoted $g(x) = Cd_f(x)$ and read "Conditional function of x knowing $f(x)$".The composite map of f and its conditional function $[f \circ Cd_f(x)) = k(x)]$ is called a transformation. $k(x)$, an elementary function, is the characteristic of transformation depending on the properties of f.*

Properties 1.2. P1) Conditional function (Cd_f) are expressed only from those of a predefined function f. It may be bijective, one-to-one or onto function; P2) If f is bijective, the conditional function $g(x) = Cd_f(x)$ is its inverse(2.1); P3) A transformation associates an infinite number of functions and depends on the domain, the number of turning points of f.

The conditional function is applied for a set of algebraic transformations with a potential interest in the determination of analytic solutions for a number of algebraic equations describing the dynamics of natural phenomena. Its properties are discussed in this paper.

2 Transformation 0 of Bijective Function

Theorem 2.1. *Let $f : A \longrightarrow B$ be a given function. If f is bijective and $g : B \longrightarrow A$ is the inverse of f, then g is also a conditional function where $g = Cd_f$. Analytic solution of $y = f(x)$ is $x = Cd_f(y)$. The composite map of f and its conditional function is a transformation defined by:*

$$f \circ Cd_f(y) = y \text{ and } Cd_f \circ f(x) = x \tag{2.1}$$

Proof Let f : $A \longrightarrow B$ be bijective. We will define a function $g : B \longrightarrow A$ as follows. Let $y \in B$. Since f is surjective, there exists $x \in A$ such that $f(x) = y$. Let $g(y) = x$. Since f is injective, this is a unique, so g is well-defined. Now we must check that g is the inverse of f. First we will show that $g \circ f = 1_A$. Let $x \in A$. Let $y = f(x)$. Then, by definition,

$$g(y) = x \text{ and } g \circ f(x) = g(f(x)) = g(y) = x \tag{2.2}$$

Now we will show that $f \circ g = 1_B$. Let $y \in B$. Let $x = g(y)$. Then, by definition,

$$f(x) = y \text{ and } f \circ g(y) = f(g(y)) = f(x) = y \tag{2.3}$$

In addition, $g = Cd_f$ by definition. From (2.2) and (2.3), analytic solution of $y = f(x)$ is $x = Cd_f(y)$ and a transformation is $f \circ Cd_f(y) = y$ or $Cd_f \circ f(x) = x$

3 Applications 1

Let $f : \Re \longmapsto \Re$ such that $xe^x = y$. This function is the product of two elementary functions, each defined on the real numbers, and each being one-to-one; but the product is not injective. Consequently, if we restrict the domain to $f : \Re^{*+} \longmapsto \Re^{*+}$, then xe^x will possess an inverse, which is a function, and it's this function that is now known as the (principal) Lambert W function [3]. It can be transformed in different ways in this domain by $x + lnx = lny$ or $e^t + t = lny$ with $x = ln(t)$ or $t^t = e^y$ with $x = e^t \cdots$. If we apply (2.2) $Cd_f(e^x + x) = x$, $Cd_f(x + lnx) = lnx$... or $e^{Cd_f(x)} + Cd_f(x) = x$, $Cd_f(x) + ln[Cd_f(x)] = x$... The unique solution expressed by an infinite number of conditonal function such that in the case of [11] is

$$x = Cd_p(y) = e^{Cd_f(lny)} = Cd_h(lny) = ln[Cd_q(e^y)] \cdots \tag{3.1}$$

where $W_0(y) = Cd_p(y)$ is the first branch of Omega function; $p(x) = xe^x$; $f(x) = e^x + x$; $h(x) = x + lnx$ and $q(x) = x^x$ are the associated functions.

4 Transformation 1 of Non Bijective Function

Theorem 4.1. *Let $f : \Re \longrightarrow [f(\varphi) \quad +\infty)$ be a given non bijective function whose derivative f' admits one turning point φ with a concave upwards curve. f is associated to an infinite number of conditional functions, $g_n = [(Cd)_n]_f$ and $g_m = [(Cd)_m]_f : \Re \longrightarrow \Re$, n and $m \in 1, 2, 3 \cdots$. The composite map of f and its main conditional function $[g_1 = [(Cd)_1]_f : \Re \longrightarrow \Re]$, $(n = 1)$, is the main or first transformation defined by*

$$f \circ [(Cd)_1]_f(x) = x^2 + f(\varphi) \tag{4.1}$$

Two consecutive conditional functions are closed by

$$[Cd_n]_f[x] = [Cd_{n+1}]_f[\sqrt{ln(x^2+1)}] \text{ and } [Cd_m]_f[x] = [Cd_{m+1}]_f[\sqrt{e^{x^2}-1}] \tag{4.2}$$

From (4.1) *and* (4.2), *secondary transformations are*

$$\begin{cases} f \circ [(Cd)_n]_f(x) = [e^{[e^{\cdot^{\cdot^{(e^{x^2}-1)}}-\cdot-\cdot-1]}} - 1] + f(\varphi) \\ f \circ [(Cd)_m]_f(x) = ln[ln(\cdots\cdots(ln(x^2+1)) + \cdots\cdots + 1) + 1] + f(\varphi) \end{cases} \tag{4.3}$$

The two analytic solutions of $y = f(x)$ *are expressed by*

$$\begin{aligned} x_{1.2} &= [Cd_m]_f(\pm\sqrt{e^{(e^{\cdot^{\cdot^{(e^{y-f(\varphi)}-1)}}-\cdot-\cdot-\cdot-1)}} - 1}) \\ &= [Cd_n]_f(\pm\sqrt{ln[(ln[......(ln[y - f(\varphi) + 1] + 1) + ... + 1] + 1) + 1]}) \\ &= [Cd_1]_f(\pm\sqrt{y - f(\varphi)}) \end{aligned}$$

Proof Let $f : \Re \longrightarrow [f(\varphi) \quad +\infty[$ be a given function. f is bijective if it is both injective and surjective. f is injective if $\forall(x_1, x_2) \in \Re^2, f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. In this case $f^{'}$ admits one turning point φ, then $\exists(x_1, x_2) \in \Re^2(x_1 < \varphi < x_2)$ such that $f(x_1) = f(x_2) \Rightarrow \mathrm{x}_1 \neq x_2$. So f is a non injective function and then non bijective. By definition, f is related to a conditional function in a way that

$$f \circ Cd_f(x) = U(x) \tag{4.4}$$

In that condition, f has two branches $f_1 : (-\infty \quad \varphi] \longrightarrow [f(\varphi) \quad +\infty)$ and $f_2 : [\varphi \quad +\infty) \longrightarrow [f(\varphi) \quad +\infty)$ whose are both bijective because $\forall[x_1 \in (-\infty \quad \varphi]$ or $x_2 \in [\varphi \quad +\infty)]$, $\exists!(y_1, y_2) \in ([f(\varphi) \quad +\infty[)^2$ such that $y_1 = f_1(x_1)$ or $y_2 = f_2(x_2)$, respectively . In addition U (an elementary function) will have the same domain with f and also one turning point. Assuming that $U(x) = x^2 + b$, if we make a changing of variable $U(x) = x^2 + b = t$, this leads to $x = \pm\sqrt{t-b}$ and $f \circ Cd_f[\pm\sqrt{t-b}] = t$

The two reciprocal branches of f_1 and f_2 are $Cd_{f_1} : [f(\varphi) \quad +\infty) \longrightarrow (-\infty \quad \varphi]$ and $Cd_{f_2} : [f(\varphi) \quad +\infty) \longrightarrow [\varphi \quad +\infty)$. For these conditions, $b = f(\varphi)$ and (4.4) becomes

$$f \circ Cd_f[\pm\sqrt{t - f(\varphi)}] = t \tag{4.5}$$

According to (2.1) and by the way that f has two branches f_1 and f_2, $Cd_f[\pm\sqrt{t-f(\varphi)}]$ of (4.5) are one of the two branches of the reciprocal function f_1 and f_2.

In addition, from (4.5), $V_1(x) = t - f(\varphi) > 0$, it exists an infinite number of in-equations satisfying $t > f(\varphi)$. If we take the exponential of both sides, the in-equation becomes $V_2(x) = e^{t-f(\varphi)} - 1 > 0$. Repeating m time the principle leads to

$$V(x)_m = e^{(e^{\cdot^{\cdot^{(e^{t-f(\varphi)}-1)}}-\cdot-\cdot-\cdot-1)}} - 1 > 0 \tag{4.6}$$

By definition,

$$f \circ [Cd_1]_f[\pm\sqrt{t - f(\varphi)}] = f \circ [Cd_2]_f[\pm\sqrt{e^{t-f(\varphi)} - 1}] = ...f \circ [Cd_m]_f[\pm\sqrt{e^{(e^{\cdot^{\cdot^{(e^{t-f(\varphi)}-1)}}-\cdot-\cdot-\cdot-1)}} - 1}] = t$$

Because $[Cd_m]_f[\pm\sqrt{V(x)_m}]$ are the inverse of the two branches of f

$$[Cd_1]_f[\pm\sqrt{t-f(\varphi)}] = [Cd_2]_f[\pm\sqrt{e^{t-f(\varphi)}-1}] = ...[Cd_m]_f[\pm\sqrt{e^{(e^{\cdot^{\cdot^{(e^{t-f(\varphi)}-1)}}}-\cdot-\cdot-\cdot-1)}-1}]$$

if we substitute $t-f(\varphi)$ by x^2,the previous equation satisfying (4.2) becomes

$$[Cd_1]_f[x] = [Cd_2]_f[\sqrt{e^{x^2}-1}] = ...[Cd_m]_f[\sqrt{e^{(e^{\cdot^{\cdot^{(e^{x^2}-1)}}}-\cdot-\cdot-\cdot-1)}-1}]$$

In other way, from (4.5), $V_1(x) = t-f(\varphi) > 0$. If we take logarithm of the both sides, in-equation becomes $V_2(x) = ln[t-f(\varphi)+1] > 0$. Repeating n time the principle leads to

$$V(x)_n = ln[(ln[......(ln[x-f(\varphi)+1]+1)+......+1]+1)+1] > 0 \tag{4.7}$$

By definition,

$$f\circ[Cd_1]_f[\pm\sqrt{t-f(\varphi)}] = ...f\circ[Cd_n]_f[\pm\sqrt{ln[(ln[...(ln[x-f(\varphi)+1]+1)+...+1]+1)+1]}] = t$$

Because $[Cd_n]_f[\pm\sqrt{V(x)_n}]$ are the inverse of the two branches of f

$$[Cd_1]_f[\pm\sqrt{t-f(\varphi)}] = ...[Cd_n]_f[\pm\sqrt{ln[(ln[...(ln[x-f(\varphi)+1]+1)+...+1]+1)+1]}]$$

if we substitute $t-f(\varphi)$ by x^2,the previous equation satisfying (4.4) becomes

$$[Cd_1]_f[x] = [Cd_2]_f[\sqrt{ln(x^2+1}] = ...[Cd_n]_f[\sqrt{ln[(ln[...(ln[x^2+1]+1)+...+1]+1)+1]}]$$

5 Transformation 2 of Non Bijective Function

Theorem 5.1. *Let $f:\Re\longrightarrow[f(\varphi_1)\quad\varphi_2[$ be a given non bijective function whose derivative f' admits one turning point φ_1 with a concave upwards curve. f is associated to an infinite number of conditional functions, $g_n = [(Cd)_n]_f$ and $g_m = [(Cd)_m]_f:\Re\longrightarrow\Re$, n and $m\in 1,2,3\cdots$. The composite map of f and its main conditional function $[g_1 = [(Cd)_1]_f:\Re\longrightarrow\Re]$, $(n=1)$, is the main or first transformation defined by*

$$f\circ[Cd_1]_f(x) = \varphi_2-[\varphi_2-f(\varphi_1)]e^{-x^2} \tag{5.1}$$

Two consecutive conditional functions also satisfied relation (4.2). From (5.1) and (4.2), secondary transformations are

$$\begin{cases} f\circ[(Cd)_n]_f(x) = \varphi_2-[\varphi_2-f(\varphi_1)]e^{-[e^{[e^{\cdot^{\cdot^{(e^{x^2}-1)}}}-\cdot-\cdot-1]}-1]} \\ f\circ[(Cd)_m]_f(x) = \varphi_2-[\varphi_2-f(\varphi_1)]e^{-ln[ln(\cdots\cdots(ln(x^2+1))+\cdots\cdots+1)+1]} \end{cases} \tag{5.2}$$

The two analytic solutions of $y=f(x)$ are

$$\begin{aligned} x_{1.2} &= [Cd_m]_f(\pm\sqrt{e^{(e^{\cdot^{\cdot^{(e^{ln[\frac{\varphi_2-f(\varphi_1)}{\varphi_2-y}]}-1)}}}-\cdot-\cdot-\cdot-1)}-1}) \\ &= [Cd_n]_f(\pm\sqrt{ln[(ln[......(ln[\frac{\varphi_2-f(\varphi_1)}{\varphi_2-y}]+1)+......+1]+1)+1]}) \\ &= [Cd_1]_f(\pm\sqrt{ln[\frac{\varphi_2-f(\varphi_1)}{\varphi_2-y}]}) \end{aligned}$$

Proof Let $f : \Re \longrightarrow [f(\varphi_1) \quad \varphi_2[$ be a given function. f is bijective if it is both injective and surjective. f is injective if $\forall(x_1, x_2) \in \Re^2, f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. For this study, $f^{'}$ admits one turning point φ_1, then $\exists(x_1, x_2) \in \Re^2(x_1 < \varphi_1 < x_2)$ such that $f(x_1) = f(x_2) \Rightarrow \mathrm{x}_1 \neq x_2$. So f is a non injective function and then non bijective. By definition, f is related to a conditional function in a way that.

$$f \circ Cd_f(x) = U(x) \tag{5.3}$$

Then, f has two branches $f_1 : (-\infty \quad \varphi_1] \longrightarrow [f(\varphi_1) \quad \varphi_2[$ and $f_2 : [\varphi_1 \quad +\infty) \longrightarrow [f(\varphi_1) \quad \varphi_2[$ whose are both bijective because $\forall x_1 \in (-\infty \quad \varphi_1]$ or $x_2 \in [\varphi_1 \quad +\infty)), \exists!(y_1, y_2) \in ([f(\varphi_1) \quad \varphi_2[)^2$ such that $y_1 = f_1(x_1)$ or $y_2 = f_2(x_2)$, respectively . This imply that from (5.3), $x = U^{-1}(x)$ and equation becomes

$$f \circ Cd_f[U^{-1}(x)] = x \tag{5.4}$$

According to (5.4), we assume that the inverse of f_1 and f_2 are such that $U^{-1}(x) = \pm\sqrt{-[ln(\varphi_2 - x) + k]}$ then

$$f \circ Cd_f[\pm\sqrt{-[ln(\varphi_2 - x) + k]}] = x \text{ and } Cd_f[\pm\sqrt{-[ln(\varphi_2 - f(x)) + k]}] = x \tag{5.5}$$

For $u(x) = \pm\sqrt{-[ln(\varphi_2 - x) + k]}$, to satisfy the relation $f(\varphi_1) \leq x < \varphi_2$ of (5.5), we must have

$$\varphi_2 - x > 0 \text{ and } [ln(\varphi_2 - x) + k] \leq 0$$

Then, $x < \varphi_2$ (true for hypothesis). In addition, the problem consist in determining k from (5.5) to obtain $x \geq f(\varphi_1)$. Thus, $[ln(\varphi_2 - x) + k] \leq 0$ leads to,

$$x \geq -e^{-k} + \varphi_2 = f(\varphi_1) \text{ with } k = -ln(\varphi_2 - f(\varphi_1))$$

By substituting k to (5.5) and taking $\dfrac{\varphi_2 - x}{\varphi_2 - f(\varphi_1)} = e^{-t^2}$, we get

$$f \circ [Cd_1]_f[\pm\sqrt{-ln(\frac{\varphi_2 - x}{\varphi_2 - f(\varphi_1)})}] = x \text{ and } f \circ [Cd_1]_f(t) = \varphi_2 - [\varphi_2 - f(\varphi_1)]e^{-t^2} \tag{5.6}$$

which is the main transformation.

In addition, from (5.6), $V(x)_1 = ln[\dfrac{\varphi_2 - f(\varphi_1)}{\varphi_2 - x}] > 0$, it exists an infinite number of in-equation satisfying $f(\varphi_1) \leq x < \varphi_2$. If we take the exponential of both sides, the in-equation becomes $V_2(x) = e^{ln[\frac{\varphi_2 - f(\varphi_1)}{\varphi_2 - x}]} - 1 > 0$. Repeating m time the principle leads to

$$V(x)_m = e^{(e^{\cdot^{\cdot^{\cdot(e^{ln[\frac{\varphi_2 - f(\varphi_1)}{\varphi_2 - x}]}-1)}}-\cdot-\cdot-\cdot-1)}} - 1 > 0 \tag{5.7}$$

By definition,

$$f \circ [Cd_1]_f[\pm\sqrt{ln[\frac{\varphi_2 - f(\varphi_1)}{\varphi_2 - x}]}] = ...f \circ [Cd_m]_f[\pm\sqrt{e^{(e^{\cdot^{\cdot^{\cdot(e^{ln[\frac{\varphi_2 - f(\varphi_1)}{\varphi_2 - x}]}-1)}}-\cdot-\cdot-\cdot-1)}} - 1}] = t$$

Because $[Cd_m]_f[\pm\sqrt{V(x)_m}]$ are the inverse of the two branches of f

$$[Cd_1]_f[\pm\sqrt{ln[\frac{\varphi_2 - f(\varphi_1)}{\varphi_2 - x}]}] = [Cd_2]_f[\pm\sqrt{e^{ln[\frac{\varphi_2 - f(\varphi_1)}{\varphi_2 - x}]} - 1}] = ...$$

if we substitute $ln[\frac{\varphi_2 - f(\varphi_1)}{\varphi_2 - x}]$ by t^2,the previous equation becomes

$$[Cd_1]_f[t] = [Cd_2]_f[\sqrt{e^{t^2} - 1}] = ...[Cd_m]_f[\sqrt{e^{(e^{\cdot^{\cdot^{(e^{t^2}-1)}}}-\cdot-\cdot-\cdot-1)} - 1}]$$

In other way, from (4.5), $V_1(x) = ln[\frac{\varphi_2 - f(\varphi_1)}{\varphi_2 - t}] > 0$. If we take logarithm of the both sides, in-equation becomes $V_2(x) = ln[ln[\frac{\varphi_2 - f(\varphi_1)}{\varphi_2 - t}] + 1] > 0$. Repeating n time the principle leads to

$$V(x)_n = ln[(ln[...(ln[ln[\frac{\varphi_2 - f(\varphi_1)}{\varphi_2 - t}] + 1] + 1) + ... + 1] + 1) + 1] > 0 \tag{5.8}$$

By definition,

$$f\circ[Cd_1]_f[\pm\sqrt{ln[\frac{\varphi_2 - f(\varphi_1)}{\varphi_2 - t}]}] = ...f\circ[Cd_n]_f[\pm\sqrt{ln[(ln[...(ln[ln[\frac{\varphi_2 - f(\varphi_1)}{\varphi_2 - t}] + 1] + 1) + ... + 1] + 1) + 1]}] = t$$

Because $[Cd_n]_f[\pm\sqrt{V(x)_n}]$ are the inverse of the two branches of f

$$[Cd_1]_f[\pm\sqrt{ln[\frac{\varphi_2 - f(\varphi_1)}{\varphi_2 - t}]}] = ...[Cd_n]_f[\pm\sqrt{ln[(ln[...(ln[ln[\frac{\varphi_2 - f(\varphi_1)}{\varphi_2 - t}] + 1] + 1) + ... + 1] + 1) + 1]}]$$

if we substitute $ln[\frac{\varphi_2 - f(\varphi_1)}{\varphi_2 - y}]$ by x^2,the previous equation becomes

$$[Cd_1]_f[x] = [Cd_2]_f[\sqrt{ln(x^2 + 1}] = ...[Cd_n]_f[\sqrt{ln[(ln[......(ln[x^2 + 1] + 1) + ... + 1] + 1) + 1]}]$$

6 Transformation 3 of Non Bijective Function

Theorem 6.1. *Let $f : \Re \longrightarrow]\varphi_1 \quad f(\varphi_2)]$ be a given non bijective function whose derivative f' admits one turning point φ_2 with a concave downwards curve. f is associated to an infinite number of conditional functions, $g_n = [(Cd)_n]_f$ and $g_m = [(Cd)_m]_f : \Re \longrightarrow \Re$, n and $m \in 1, 2, 3 \cdots$. The composite map of f and its main conditional function $[g_1 = [(Cd)_1]_f : \Re \longrightarrow \Re]$, $(n = 1)$, is the main or first transformation expressed by*

$$f \circ [(Cd)_1]_f(x) = \varphi_1 + [f(\varphi_2) - \varphi_1]e^{-x^2} \tag{6.1}$$

Two consecutive conditional functions also satisfied relation (4.2). The secondary transformations deduced from (6.1) and (4.2) are

$$\begin{cases} f \circ [(Cd)_n]_f(x) = \varphi_1 + [f(\varphi_2) - \varphi_1]e^{-[e^{[e^{\cdot^{\cdot^{(e^{x^2}-1)}}}-\cdot-\cdot-1]}-1]} \\ f \circ [(Cd)_m]_f(x) = \varphi_1 + [f(\varphi_2) - \varphi_1]e^{-ln[ln(\cdots\cdots(ln(x^2+1))+\cdots\cdots+1)+1]} \end{cases} \tag{6.2}$$

The two analytic solutions of $y = f(x)$ *are*

$$
\begin{aligned}
x_{1.2} &= [Cd_m]_f(\pm\sqrt{e^{(e^{\cdot^{\cdot^{\cdot(e^{ln[\frac{f(\varphi_2)-\varphi_1}{y-\varphi_1}]}-1)}}}-\cdot-\cdot-\cdot-1)}-1}) \\
&= [Cd_n]_f(\pm\sqrt{ln[(ln[......(ln[\frac{f(\varphi_2)-\varphi_1}{y-\varphi_1}]+1)+......+1]+1)+1]}) \\
&= [Cd_1]_f(\pm\sqrt{ln[\frac{f(\varphi_2)-\varphi_1}{y-\varphi_1}]})
\end{aligned}
$$

Proof Let $f : \Re \longrightarrow]\varphi_1 \quad f(\varphi_2)]$ be a given function. $f^{'}$ admits one turning point φ_2,with a concave downwards curve, and is a non bijective function as proved for the previous function. By definition, f is related to a conditional function in a way that

$$f \circ Cd_f(x) = U(x) \tag{6.3}$$

In that condition, U (an elementary function) will have the same domain with f and also one turning point. So, f has also two branches f_1 and f_2 whose are both bijective. This imply that from (6.3), $x = U^{-1}(x)$ and equation becomes

$$f \circ Cd_f[U^{-1}(x)] = x \tag{6.4}$$

According to (6.4), we assume that the inverse of f_1 and f_2 are such that $U^{-1}(x) = \pm\sqrt{-[ln(x-\varphi_1)+k]}$ then

$$f \circ [Cd_1]_f[\pm\sqrt{-[ln(x-\varphi_1)+k]}] = x \text{ and } [Cd_1]_f[\pm\sqrt{-[ln(f(x)-\varphi_1)+k]}] = x \tag{6.5}$$

Let $U^{-1}(x) = \pm\sqrt{-[ln(x-\varphi_1)+k]}$, to satisfy the relation $\varphi_1 < x \leq f(\varphi_2)$ of (6.5), we must have

$$x - \varphi_1 > 0 \text{ and } [ln(x-\varphi_1)+k] \leq 0$$

Then, $x > \varphi_1$ (true for hypothesis). In addition, the problem consist in determining k from (6.5) to obtain $x \leq f(\varphi_2)$. Thus, $ln(x-\varphi_1)+k \leq 0$ leads to,

$$x \leq e^{-k} + \varphi_1 = f(\varphi_2) \text{ with } k = -ln(f(\varphi_2)-\varphi_1)$$

By substituting k to (6.5) and taking $\frac{f(\varphi_2)-\varphi_1}{x-\varphi_1} = e^{t^2}$, we get

$$f \circ [Cd_1]_f[\pm\sqrt{ln[\frac{f(\varphi_2)-\varphi_1}{x-\varphi_1}]}] = x \text{ and } f \circ [Cd_1]_f(t) = \varphi_1 + [f(\varphi_2)-\varphi_1]e^{-t^2} \tag{6.6}$$

which is the main transformation. Many other infinite functions are related to f and are also proved as in the case of transformation 2.

7 Transformation 4 of Non Bijective Function

Theorem 7.1. *Let* $f : \Re \longrightarrow (-\infty \quad f(\varphi)]$ *be a given non bijective function whose derivative* $f^{'}$ *admits one turning point* φ *with a concave downwards curve.* f *is associated to an infinite number of conditional functions,* $g_n = [(Cd)_n]_f$ *and* $g_m = [(Cd)_m]_f : \Re \longrightarrow \Re$*,* n *and* $m \in 1, 2, 3\cdots$*. The composite map of* f *and its main conditional function* $[g_1 = [(Cd)_1]_f : \Re \longrightarrow \Re]$*,* $(n = 1)$*, is the main transformation expressed by*

$$f \circ [(Cd)_1]_f(x) = f(\varphi) - x^2 \tag{7.1}$$

Two consecutive conditional functions also satisfied relation (4.2). *The secondary transformations deduced from* (7.1) *and* (4.2) *are*

$$\begin{cases} f \circ [(Cd)_n]_f(x) = f(\varphi) + 1 - e^{[e^{\cdot^{\cdot^{(e^{t^2}-1)}}-\cdot-\cdot-1]}} \\ f \circ [(Cd)_m]_f(x) = f(\varphi) - ln[ln(\cdots\cdots(ln(x^2+1)) + \cdots\cdots + 1) + 1] \end{cases} \quad (7.2)$$

The two analytic solutions of $y = f(x)$ are

$$\begin{aligned} x_{1.2} &= [Cd_m]_f(\pm\sqrt{e^{(e^{\cdot^{\cdot^{(e^{f(\varphi)-y}-1)}}-\cdot-\cdot-\cdot-1)}} - 1}) \\ &= [Cd_n]_f(\pm\sqrt{ln[(ln[......(ln[f(\varphi)+1-y]+1) + ... + 1] + 1) + 1]}) \\ &= [Cd_1]_f(\pm\sqrt{f(\varphi) - y}) \end{aligned}$$

Proof Let $f : \Re \longrightarrow (-\infty \quad f(\varphi)]$ be a given function. f admits one turning point φ, with a concave downwards curve, then $\exists(x_1, x_2) \in \Re^2(x_1 < \varphi < x_2)$ such that $f(x_1) = f(x_2) \Rightarrow \mathrm{x}_1 \neq x_2$. So f is a non bijective function. Since, f has two branches f_1 and f_2 whose are both bijective because $\forall[x_1 \in (-\infty \quad \varphi]$ or $x_2 \in [\varphi \quad +\infty)]$, $\exists!(y_1, y_2) \in ([f(\varphi) \quad +\infty[)^2$ such that $y_1 = f_1(x_1)$ or $y_2 = f_2(x_2)$, respectively . In addition U and f have the same domain and one turning point each. Assuming that $U(x) = -x^2 + b$ is a polynomial function, if we make a changing of variable $U(x) = -x^2 + b = t$, this leads to $x = \pm\sqrt{b - t}$ and $f \circ Cd_f[\pm\sqrt{b - t}] = t$

The two reciprocal branches of f_1 and f_2 are $Cd_{f_1} : (-\infty \quad f(\varphi)] \longrightarrow (-\infty \quad \varphi]$ and $Cd_{f_2} : (-\infty \quad f(\varphi] \longrightarrow [\varphi \quad +\infty)$. For these conditions, $b = f(\varphi)$ and

$$f \circ Cd_f[\pm\sqrt{f(\varphi) - t}] = t \quad (7.3)$$

According to (2.1) and by the way that f has two branches f_1 and f_2, $Cd_f[\pm\sqrt{f(\varphi) - t}]$ of (7.3) are one of the two branches of the reciprocal function of f_1 and f_2.

In addition, from (7.3), $V_1(x) = f(\varphi) - t > 0$, it exists an infinite number of in-equation satisfying $t < f(\varphi)$. If we take the exponential or the logarithm of both sides, the in-equation becomes $V_2(x) = e^{f(\varphi)-t} - 1 > 0$ or $V_2(x) = ln(f(\varphi) - t + 1) > 0$, respectively. Repeating m or n time the principle leads to $V_m(x)$ and $V_n(x)$ as in the case of transformation 1 and finally to

$$[Cd_1]_f[x] = [Cd_2]_f[\sqrt{ln(x^2 + 1}] = ...[Cd_n]_f[\sqrt{ln[(ln[...(ln[x^2 + 1] + 1) + ... + 1] + 1) + 1]}]$$

with $[Cd_m]_f[\pm\sqrt{V(x)_m}]$ and $[Cd_n]_f[\pm\sqrt{V(x)_n}]$ expressed in term of inverses of the two branches of f

8 Applications 2

Example 8.1. *Given $(ax + b)p^{(cx+d)} + gx = y$, the changing of variable leads to*

$$te^t + nt = Y \text{ with } x = \frac{t}{clnp} - \frac{b}{a} \quad (8.1)$$

The determinant (n) is

$$n = \frac{gp^{(bc/a)-d}}{a} \text{ with } Y = [(y + \frac{bg}{a})\frac{clnp}{a}]p^{(bc/a)-d} \quad (8.2)$$

Contrary to the conditional function, analytic solutions of the reduced form $te^t + nt = Y$ and its conjugate $te^t - nt = \overline{Y}$ are partially expressed by the Omega function when $n = 0$. The generalisation belongs to the different classes of functions deriving from the conditional function.

$$\begin{cases} f_n(t) = te^t + nt, \text{ for } n = 0, n > 1/e^2, n = 1/e^2, 0 < n < 1/e^2 \text{ and } n < 0 \\ f_n(t) = te^t - nt, \text{ for } n = 0, n < -1/e^2, n = -1/e^2, -1/e^2 < n < 0, \text{ and } n > 0 \end{cases} \tag{8.3}$$

(i) *If $n = 0$, and $0 < Y < +\infty$, then $f_0(t) = \overline{f}_0(t)$. From (2.1), $Cd_{f_{0_1}}(Y)e^{Cd_{f_{0_1}}(Y)} = Y$ where $Cd_{f_{0_1}} = W_{0_1}$ is the first branch of Omega function and $t = Cd_{f_{0_1}}(Y)$. Final solution is expressed*

$$x = \frac{Cd_{f_{0_1}}([\frac{(yc)lnp}{a}]p^{(\frac{bc}{a}-d)})}{clnp} - \frac{b}{a} \tag{8.4}$$

(ii) *If $n = 0$, and $(\frac{-1}{e} \leq Y < 0 = \varphi_2$ and $x < 0)$, the derivative $(t+1)e^t$ cancel at $t = -1$. By (5.2), $Cd_{f_{0_2}}(Y)e^{Cd_{f_{0_2}}(Y)} = Y$ where $Cd_{f_{0_2}} = W_{0_2}$ is the second branch of Omega function. Then,*

$$\begin{aligned} t_{1,2} &= W_{0_2}(Y) \\ &= Cd_{h_{0_2}}(\pm\sqrt{-[ln(-Y)+1]}) \\ &= -Cd_{k_{0_2}}(\pm\sqrt{-[ln(-Y)+1]}) \\ &= -e^{Cd_{g_{0_2}}(\pm\sqrt{-[ln(-Y)+1]})} \end{aligned}$$

with $h(t) = te^t$; $k(t) = t - lnt$, $g(t) = e^t - t$. We noticed that $W_{0_2}(0)$ is not defined in $\Re$ as described by [3]

(iii) *If $n < 0$, and $-2n + n[e^{-Cd_g(ln[-ne])} + e^{Cd_g(ln[-ne])}] \leq Y < +\infty$, the derivative $(t+1)e^t = n$ cancel at $t = e^{Cd_{g_{0_1}}(ln[-ne])} - 1$. Using (4.1),*

$$Cd_{f_n}(Y)(e^{Cd_{f_n}(Y)} + n) = Y^2 - 2n + n[e^{-Cd_g(ln[-ne])} + e^{Cd_g(ln[-ne])}] \tag{8.5}$$

(with $t = Cd_{f_n}(Y)$). The two final solutions are

$$x = \frac{Cd_{f_n}(\pm\sqrt{[(y+\frac{bg}{a})\frac{clnp}{a}]p^{(\frac{bc}{a}-d)} + 2n - n[e^{-Cd_g(ln[-ne])} + e^{Cd_g(ln[-ne])}]})}{clnp} - \frac{b}{a} \tag{8.6}$$

(iv) *If $n > 1/e^2$, and $Y \in \Re$, the derivative $(t+1)e^t = n$ is strictly positive in a set $\Re$. From (2.1), $Cd_{f_n}(Y)(e^{Cd_{f_n}(Y)} + n) = Y$ (with $t = Cd_{f_n}(Y)$). The final solution is expressed*

$$x = \frac{Cd_{f_n}([(y+\frac{bg}{a})\frac{clnp}{a}]p^{(\frac{bc}{a}-d)})}{clnp} - \frac{b}{a} \tag{8.7}$$

(v) *If $n = 1/e^2$, and $Y \in \Re$, the derivative $(t+1)e^t = n$ cancel at $t = 1$ is strictly positive in a set $\Re$ and there is an inflexion point at $x = -2$; then (2.1) is applied.*

(vi) *If $0 < n < 1/e^2$, and $Y \in \Re$, the derivative $(t+1)e^t = n$ cancel at*

$$t_{1,2} = \varphi_{1,2} = -e^{Cd_{g_n}(\pm\sqrt{-ln[-ne^2]})} - 1.$$

By applying (9.19), *there are three sub classes which should be defined in the next section.*

$$Cd_{f_n}(Y)(e^{Cd_{f_n}(Y)}+n)=\begin{cases} f_n(\varphi_1)-e^Y \\ e^Y+f_n(\varphi_2) \\ Y^3+\frac{3}{2}[f(\varphi_2)-f(\varphi_1)]^{1/3}Y+\frac{f(\varphi_2)+f(\varphi_1)}{2} \end{cases} \tag{8.8}$$

Remark 8.2. Given $y+ln(y)=z=x+iy$, the unique solution is $y=Cd_f(x+iy)$ $[(x,y)\in\Re^2]$. The [4] function was developed and denoted ω. It is defined in terms of the Lambert W function as: $\omega(z)=W_{[(Im(z)-\pi)/2\pi]}(e^z)$. Wright showed that $y=\omega(z)$ is the unique solution, when $z\neq x\pm i\pi$ for $x\leq -1$, of the equation $y+ln(y)=z$. Contrary to Cd_f, Wright function is not continuous in the domain. In addition, $W_0\neq 0$ is contradictory to the study of [3]. In contrast, $W_0(x)$ approaches 0 when x approaches 0 according to the Lagrange inversion theorem [13]. $xe^x=0$ at that point leads to the solution

$$x=0 \quad (e^x\neq 0) \tag{8.9}$$

instead of $x=w_0(0)=0$

Example 8.3. *Given* $p^{(ax^2+bx+c)}+(dx^2+fx+g)=0$ *defined in* $\Re$, *with* $[(a,d)\neq(0,0)]$.

(i) $a>0$ *and* $p>1$ *or* $a<0$ *and* $0<p<1$*: Two reduced forms are*

$$e^{t^2}+n(t+m)^2=Y \text{ and } e^{t^2}+n(m-t)^2=Y \tag{8.10}$$

with

$$Y=\frac{f^2-4dg}{4d}[p]^{\frac{(b^2-4ac)}{4a}} \text{ and } n=\frac{d}{aln(p)}[p]^{\frac{(b^2-4ac)}{4a}} \text{ (first determinant)} \tag{8.11}$$

For each of the reduced form, the second determinant is

$$m=(f-\frac{bd}{a})\frac{\sqrt{aln(p)}}{2d} \text{ and } m=(\frac{bd}{a}-f)\frac{\sqrt{aln(p)}}{2d} \tag{8.12}$$

respectively and

$$x=\frac{\pm t}{\sqrt{aln(p)}}-\frac{b}{2a} \tag{8.13}$$

The two derivatives of reduced forms are $2[t(e^{t^2}+n)+nm]$ *and* $2[t(e^{t^2}+n)-nm]$ *respectively. They have a same second derivative* $Y^{''}=2[(1+2t^2)e^{t^2}]+n$. *In these conditions when* $n<-1$, $Y^{''}$ *has two turning points*

$$\varphi_{1.2}=\pm\sqrt{e^{Cd_f(ln[-\frac{ne^{1/2}}{2}])}-1/2} \tag{8.14}$$

Particularly when $n=-1$, $Y^{''}$ *has one turning point* $\varphi_1=0$ *and when* $-1<n<0$ *then* $Y^{''}>0$. *In addition when* $n>0$, $Y^{''}>0$.*These initial conditions show that generalisation belongs to different classes of functions*

Case 8.1. $n<0$ *and* $m=0$, $e^{t^2}+n(t)^2=Y$ *and referring to* (4.3)

$$t=\pm\sqrt{Cd_g(\pm\sqrt{-\frac{Y}{n}-ln(-\frac{1}{n})-1})-ln(-\frac{1}{n})} \tag{8.15}$$

Case 8.2. $n > 0$ *and* $m = 0$, *by applying* (2.1) *to* $e^{t^2} + n(t)^2 = Y$,

$$t = \pm\sqrt{Cd_f[\frac{Y}{n} + ln(\frac{1}{n})] - ln(\frac{1}{n})} \tag{8.16}$$

Case 8.3. $n > 0$ *and* $m \neq 0$ *and* $e^{t^2} + n(t+m)^2 = Y$, *the derivative of* $Y^{'}$ *is* $2[(1+2t^2)e^{t^2}] + n > 0$. *Then,* $Y^{'}$ *is bijective and* $Cd_{g_n}[t(e^{t^2} + n)] = Cd_{g_n}[-nm]$ *leading to* $t = Cd_{g_n}[-nm]$ *with* $\varphi = Cd_{g_n}[-nm]$ *which is the turning point of* y. *Referring to* (4.3)

$$t = Cd_{f_{m.n}}[\pm\sqrt{Y - e^{[Cd_{g_n}[-m.n]]^2} - n(Cd_{g_n}[-m.n] + m)^2}] \tag{8.17}$$

where $f_{m.n} = e^{t^2} + n(t+m)^2$ *and* $g_n = t(e^{t^2} + n)$

Case 8.4. $n > 0$ *and* $m \neq 0$ *and* $e^{t^2} + n(m-t)^2 = Y$, *the derivative of* $y^{'}$ *is* $2[(1+2t^2)e^{t^2}] + n > 0$. *Then,* $Y^{'}$ *is bijective and* $Cd_{g_n}[t(e^{t^2} + n)] = Cd_{g_n}[nm]$ *leading to* $t = Cd_{g_n}[nm]$ *with* $\varphi = Cd_{g_n}[nm]$ *which is the turning point of* y. *By* (4.3),

$$t = Cd_{h_{m.n}}[\pm\sqrt{Y - e^{[Cd_{g_n}[m.n]]^2} - n(m - Cd_{g_n}[m.n])^2}] \tag{8.18}$$

where $h_{m.n} = e^{t^2} + n(m-t)^2$ *and* $g_n = t(e^{t^2} + n)$

Case 8.5. $n < -1$ *and* $m \neq 0$ *and* $e^{t^2} + n(t+m)^2 = Y$, *derivatives of* Y *are*

$$Y^{'} = 2[t(e^{t^2} + n) + nm] \text{ and } Y^{''} = 2[(1+2t^2)e^{t^2}] + n \tag{8.19}$$

$Y^{''}$ *has two turning points*

$$\varphi_{1.2} = \pm\sqrt{e^{Cd_f(ln[-\frac{ne^{1/2}}{2}])} - 1/2}, (\text{ with } n \leq -1) \tag{8.20}$$

$*$ *If* $-\infty < -nm < f^{'}_{mn}(\varphi_1)$,$Y^{'}$ *has one turning point* $\varphi_1 = Cd_{f^{'}_{nm}}[ln(f^{'}_{mn}(\varphi_1)) + nm)]$ *and* (4.3) *is applied to* Y

$$t = Cd_{f_{m.n}}[\pm\sqrt{Y - e^{[Cd_{f'_{nm}}[ln(f'_{mn}(\varphi_1)) - nm)]]^2} - n(Cd_{f'_{nm}}[ln(f'_{mn}(\varphi_1)) - nm)] + m)^2}]$$

$*$ *If* $f^{'}_{mn}(\varphi_2) < -nm < +\infty$, $Y^{'}$ *has one turning point* $\varphi_2 = Cd_{f^{'}_{nm}}[-nm - ln(f^{'}_{mn}(\varphi_2)))]$ *and referring to* (4.3),

$$t = Cd_{f_{m.n}}[\pm\sqrt{Y - e^{[Cd_{f'_{nm}}[-nm - ln(f'_{mn}(\varphi_2)))]]^2} - n(Cd_{f'_{nm}}[-nm - ln(f'_{mn}(\varphi_2)))] + m)^2}]$$

$*$ *If* $f^{'}_{mn}(\varphi_1) < -nm < f^{'}_{mn}(\varphi_2)$, $Y^{'}$ *has three turning points*

$$\varphi_k = Cd_{f'_{nm}}[Vcos[\frac{1}{3}arcos([\frac{y}{2} - \frac{f^{'}_{nm}(\varphi_1) + f^{'}_{nm}(\varphi_2)}{4}]\sqrt{\frac{8}{[f'_{nm}(\varphi_1) - f'_{nm}(\varphi_2)]}}) + \frac{2k\pi}{3}]$$

where

$$V = 2\sqrt{\frac{1}{2}[f'_{nm}(\varphi_2) - f'_{nm}(\varphi_1)]^{1/3}} and k \in \{0, 1, 2\}$$

Transformation 6 is applied and there are three subclasses of solution.

$*$ *If* $-nm = f^{'}_{mn}(\varphi_2)$ *or* $-mn = f^{'}_{mn}(\varphi_1)$, $Y^{'}$ *has two turning points.* (9.19) *is applied and there are three subclasses of solution.*

Case 8.6. *$n < -1$ and $m \neq 0$ and $e^{t^2} + n(m-t)^2 = Y$, the derivative of Y is $Y' = 2[t(e^{t^2}+n) - nm]$ and $Y'' = 2[(1+2t^2)e^{t^2}] + n$. Y'' has also two turning points*

$$\varphi_{1.2} = \pm\sqrt{e^{Cd_f(ln[-\frac{ne^{1/2}}{2}])} - 1/2}(\text{ with } n \leq -1) \tag{8.21}$$

$*$ *If $-\infty < nm < h'_{mn}(\varphi_1)$,$Y'$ has one turning point $\varphi_1 = Cd_{h'_{nm}}[ln(h'_{mn}(\varphi_1)) - nm)]$ and (4.3) is applied to Y*

$*$ *If $h'_{mn}(\varphi_2) < nm < +\infty$, Y' has one turning point $\varphi_2 = Cd_{h'_{nm}}[nm - ln(h'_{mn}(\varphi_2)))]$ and (4.3) is applied to Y.*

$*$ *If $h'_{mn}(\varphi_1) < nm < h'_{mn}(\varphi_2)$, Y' has three turning points*

$$\varphi_k = Cd_{h'_{nm}}[Ucos[\frac{1}{3}arcos([\frac{y}{2} - \frac{h'_{nm}(\varphi_1) + f(\varphi_2)}{4}]\sqrt{\frac{8}{[f'_{nm}(\varphi_1) - h'_{nm}(\varphi_2)]}}) + \frac{2k\pi}{3}] \tag{8.22}$$

where

$$U = 2\sqrt{\frac{1}{2}[h'_{nm}(\varphi_2) - h'_{nm}(\varphi_1)]^{1/3}} and k \in \{0,1,2\}$$

Transformation 6 is applied and Y has three subclasses of solutions.

$*$ *If $nm = h'_{mn}(\varphi_2)$ or $nm = h'_{mn}(\varphi_1)$, Y' has two turning points, by (9.3), Y has three subclasses of solutions.*

Case 8.7. *$n = -1$ and $m \neq 0$ then*

$$e^{t^2} - (t-m)^2 = Y \text{ or } e^{t^2} - (m-t)^2 = Y \tag{8.23}$$

for each function $Y'' = 2[(1+2t^2)e^{t^2}] - 1$. Y'' has one turning point $\varphi_1 = 0$. (9.3) is applied.

Case 8.8. *$-1 < n < 0$ and $m \neq 0$ and*

$$e^{t^2} + n(t-m)^2 = Y \text{ or } e^{t^2} - (m-t)^2 = Y \tag{8.24}$$

for each function $Y'' = 2[(1+2t^2)e^{t^2}] + n$. $Y'' > 0$ (5.2) is applied.

(ii) *$a < 0$ and $p > 1$ or $a > 0$ and $0 < p < 1$, two reduced forms are*

$$e^{-t^2} - n(t+m)^2 = Y \text{ and } e^{-t^2} - n(m-t)^2 = Y \tag{8.25}$$

with

$$Y = \frac{f^2 - 4dg}{4d}[p]^{\frac{(b^2-4ac)}{4a}} \text{ and } n = \frac{d}{aln(p)}[p]^{\frac{(b^2-4ac)}{4a}} \text{ (fist determinant)} \tag{8.26}$$

For each of the reduced form, the second determinant is

$$m = (f - \frac{bd}{a})\frac{\sqrt{-aln(p)}}{2d} \text{ and } m = (\frac{bd}{a} - f)\frac{\sqrt{-aln(p)}}{2d} \tag{8.27}$$

respectively and

$$x = \frac{\pm t}{\sqrt{-aln(p)}} - \frac{b}{2a} \tag{8.28}$$

The two derivatives of reduced forms are $-2[t(e^{-t^2}+n)+nm]$ *and* $-2[t(e^{-t^2}+n)-nm]$ *respectively. They have a same second derivative* $Y^{''}=2[(2t^2-1)e^{-t^2}-n]$ *. In these conditions when* $-1<n<0$, $Y^{''}$ *has two turning points*

$$\varphi_{1,2}=\pm\sqrt{-e^{Cd[ln(\frac{-n}{2}e^{1/2})]}+1/2} \tag{8.29}$$

Particularly when $n=-1$, $Y^{''}$ *has one turning point* $\varphi_1=0$ *and when* $n<-1$ *then* $Y^{''}>0$. *In addition when* $n>0$, $Y^{''}>0$.*These initial conditions show that generalisation belongs to different classes of functions*

Referring to (8.10), (8.25) *has also eight cases:*

Case 8.9. $n>0$ *and* $m=0$*;*

Case 8.10. $n<0$ *and* $m=0$*;*

Case 8.11. $n>0$ *and* $m\neq 0$ *and* $e^{-t^2}-n(t+m)^2=Y$*;*

Case 8.12. $n>0$ *and* $m\neq 0$ *and* $e^{-t^2}-n(m-t)^2=Y$*;*

Case 8.13. $-1<n<0$ *and* $m\neq 0$ *and* $e^{-t^2}-n(t+m)^2=Y$*;*

$*$ *If* $-\infty<-nm<f^{'}_{mn}(\varphi_1)$*;*

$*$ *If* $f^{'}_{mn}(\varphi_2)<-nm<+\infty$*;*

$*$ *If* $f^{'}_{mn}(\varphi_1)<-nm<f^{'}_{mn}(\varphi_2)$*;*

$*$ *If* $-nm=f^{'}_{mn}(\varphi_2)$ *or* $-mn=f^{'}_{mn}(\varphi_1)$, $Y^{'}$ *has two turning points. Transformation 5 of next paragraph is applied and* Y *has three subclasses of solution.*

Case 8.14. $-1<n<0$ *and* $m\neq 0$ *and* $e^{-t^2}+n(m-t)^2=Y$*;*

$*$ *If* $nm\in]-\infty \quad h^{'}_{mn}(\varphi_1)[$*;*

$*$ *If* $h^{'}_{mn}(\varphi_2)<nm<+\infty$*;*

$*$ *If* $h^{'}_{mn}(\varphi_1)<nm<h^{'}_{mn}(\varphi_2)$*;*

$*$ *If* $nm=h^{'}_{mn}(\varphi_2)$ *or* $nm=h^{'}_{mn}(\varphi_1)$*;*

Case 8.15. $n=-1$ *and* $m\neq 0$ *then*

$$e^{-t^2}-(t-m)^2=Y \text{ or } e^{-t^2}-(m-t)^2=Y \tag{8.30}$$

Case 8.16. $n<-1$ *and* $m\neq 0$ *and*

$$e^{-t^2}+n(t-m)^2=Y \text{ or } e^{-t^2}-(m-t)^2=Y \tag{8.31}$$

9 Transformation 5 of a Non Bijective Function

Let $f(x)=y$ be defined in a set $A\leq\Re$. If $f(x)^{'}$ has two turning points φ_1 and φ_2 with $\varphi_2>\varphi_1$ or $\varphi_2<\varphi_1$ depending on the sign of $f^{'}(x)$ and $f(\varphi_2)<f(\varphi_1)$ then, $-\infty<y<f(\varphi_2)$ and $f(\varphi_2)<y<f(\varphi_1)$ and $f(\varphi_1)<y<+\infty$ and $[y,h]=[f(\varphi_1),f(\varphi_2)]$. The function has three branches.

9.1 First branch of transformation

Theorem 9.1.1. *Let $f_1 : A \in \Re \longrightarrow (-\infty \quad f(\varphi_2)[$ be a branch of f. f_1 is bijective whose derivative $f_1^{'} > 0$. f_1 is associated to an infinite number of conditional functions, $g_n = [(Cd)_n]_f$ and $g_m = [(Cd)_m]_f : \Re \longrightarrow \Re$, n and $m \in 1, 2, 3 \cdots$. The composite map of f_1 and its main conditional function $[g_1 = [(Cd)_1]_f : \Re \longrightarrow \Re]$, $(n = 1)$, is the main or first transformation defined by*

$$f_1 \circ [(Cd)_1]_{f_1}(x) = f(\varphi_2) - e^x \tag{9.1}$$

Two consecutive conditional functions satisfied the relation

$$[(Cd)_n]_f(x) = [(Cd)_{n+1}]_f ln[ln(e^x + 1)] \text{ and } [(Cd)_m]_f(x) = [(Cd)_{m+1}]_f ln(e^{e^x} - 1) \tag{9.2}$$

From (9.1) and (9.2), secondary transformations are

$$\begin{cases} f_1 \circ [(Cd)_n]_{f_1}(x) = f(\varphi_2) - [e^{[e^{\cdot^{\cdot^{(e^{e^x}-1)}}}-\cdot-\cdot-1]} - 1] \\ f_1 \circ [(Cd)_m]_{f_1}(x) = f(\varphi_2) - ln[ln(\cdots\cdots(ln(e^x + 1)) + \cdots\cdots + 1) + 1] \end{cases} \tag{9.3}$$

The analytic solution of $y = f_1(x)$ is

$$\begin{aligned} x &= [(Cd)_m]_{f_1}[ln(e^{(e^{\cdot^{\cdot^{(e^{f(\varphi_2)-y}-1)}}}-\cdot-\cdot-\cdot-1)} - 1)] \\ &= [(Cd)_n]_{f_1}[ln(ln[(ln[......(ln[f(\varphi_2) - y + 1] + 1) + ... + 1] + 1) + 1])] \\ &= (Cd)_{1f_1}[ln(f(\varphi_2) - y)] \end{aligned}$$

Proof: Let $f_1 : A \in \Re \longrightarrow (-\infty \quad f(\varphi_2)[$ be bijective. We will define a function Cd_{f_1} : $(-\infty \quad f(\varphi_2)[\longrightarrow A \in \Re$. Let $y \in (-\infty \quad f(\varphi_2)[$. Since f is surjective, there exists $x \in A$ such that $f_1(x) = y$. Let $x = (Cd)_{1f_1}[ln(f(\varphi_2) - y)]$. Since f is injective, this is a unique, so Cd_{f_1} is well-defined. Now we most check that $(Cd)_{1f_1}[ln(f(\varphi_2) - y)]$ is the inverse of f_1. First we will show that $Cd_{f_1} \circ f_1 = 1_A$. Let $x \in A$. Let $y = f(x)$. Then, by definition,

$$(Cd)_{1f_1}[ln(f(\varphi_2) - y)] = x \text{ and } Cd_{f_1} \circ f_1(x) = (Cd)_{1f_1}[ln(f(\varphi_2) - f_1(x))] = x \tag{9.4}$$

Now we will show that $f \circ Cd_{f_1} = 1_B$. Let $y \in B$. Let $x = g(y)$. Then, by definition,

$$f_1(x) = y \text{ and } f_1 \circ g(y) = f_1((Cd)_{1f_1}[ln(f(\varphi_2) - y)]) = f_1(x) = y \tag{9.5}$$

In addition from $f(\varphi_2) - y > 0$ of (9.4), it exists a set of in-equations satisfying $y < f(\varphi)$. If we take the exponential or the logarithm of both sides, the in-equation becomes $V_1(x) = e^{f(\varphi_2)-y} - 1 > 0$ or $V_2(x) = ln(f(\varphi_2) - y + 1) > 0$, respectively. Repeating m or n time the principle leads to

$$V_n(x) = ln[(ln[......(ln[f(\varphi_2) - x + 1] + 1) + ... + 1] + 1) + 1] > 0 \tag{9.6}$$

$$V_m(x) = e^{(e^{\cdot^{\cdot^{(e^{f(\varphi_2)-x}-1)}}}-\cdot-\cdot-\cdot-1)} - 1 > 0 \tag{9.7}$$

Rearanging the above expression in (9.1) will give the relation between two consecutive conditional function of (9.2)

Example 9.1.2. Given $e^{-cx} = a_0(x-r_1)(x-r_2)$, it expresses the equation governing the dilaton field, from which is derived the metric of the R=T or lineal two-body gravity problem in 1+1 dimensions (one spatial dimension and one time dimension) for the case of unequal (rest) masses, as well as, the eigenenergies of the quantum-mechanical double-well Dirac delta function model for unequal charges in one dimension. The reduced form of equation is $e^t - nt^2 = y(n,t)$ where

$$y = [\frac{-(r_1+r_2)^2}{4a_0} + r_1 r_2]e^{[}\frac{-(r_1+r_2)c}{2a_0}], \tag{9.8}$$

$$x = \frac{-t}{c} + \frac{(r_1+r_2)}{2a_0} \text{ and } n = \frac{a_0}{c^2}e^{[}\frac{-(r_1+r_2)c}{2a_0}] \tag{9.9}$$

The generalization resembles the hypergeometric function but it belongs to a different class of functions (with $n > \frac{e}{2}$, $n < \frac{e}{2}$ and $n = \frac{e}{2}$) [12].

If $n > \frac{e}{2}$, y has two turning points $t = Cd_h(\pm\sqrt{ln(2n)-1}) + ln(2n)$ (with $h(t) = e^t - t$),

$$\begin{cases} f(\varphi_2) = e^{[Cd_h(-\sqrt{ln(2n)-1})+ln(2n)]} - n[Cd_h(-\sqrt{ln(2n)-1}) + ln(2n)]^2 \quad \text{and} \\ f(\varphi_1) = e^{[Cd_h(\sqrt{ln(2n)-1})+ln(2n)]} - n[Cd_h(\sqrt{ln(2n)-1}) + ln(2n)]^2 \end{cases} \tag{9.10}$$

In the set $-\infty < y < f(\varphi_2)$, the transformation is

$$e^{Cd_{f_1}(t)} - n[Cd_{f_1}(t)]^2 = e^{[Cd_h(-\sqrt{ln(2n)-1})+ln(2n)]} - n[Cd_h(-\sqrt{ln(2n)-1}) + ln(2n)]^2 - e^t \tag{9.11}$$

which yields the final and unique solution

$$x = \frac{-Cd_{f_1}[ln(e^{[Cd_h(-\sqrt{ln(2n)-1})+ln(2n)]} - n[Cd_h(-\sqrt{ln(2n)-1}) + ln(2n)]^2 - y)]}{c} + \frac{(r_1+r_2)}{2a_0}$$

9.2 Second branch of transformation

Theorem 9.2.1. *Let $f_2 : \Re \longrightarrow]f(\varphi_1) \quad +\infty)$ be a second branch of f. f_2 is bijective whose derivative $f_2' > 0$. f_2 is associated to an infinite number of conditional functions, $g_n = [(Cd)_n]_f$ and $g_m = [(Cd)_m]_f : \Re \longrightarrow \Re$, n and $m \in 1, 2, 3\cdots$. The composite map of f_2 and its main conditional function $[g_1 = [(Cd)_1]_f : \Re \longrightarrow \Re]$, $(n = 1)$, is the main or first transformation defined by*

$$f_2 \circ [(Cd)_2]_f(x) = f(\varphi_1) + e^x \tag{9.12}$$

Two consecutive conditional functions satisfied the relation of (9.2)

From (9.12) *and* (9.2)*, secondary transformations are*

$$\begin{cases} f_2 \circ [(Cd)_n]_{f_2}(x) = f(\varphi_1) + [e^{[e^{\cdot^{\cdot^{(e^{e^x}-1)}}-\cdot-\cdot-1]}} - 1] \\ f_2 \circ [(Cd)_m]_{f_2}(x) = f(\varphi_1) + ln[ln(\cdots\cdots(ln(e^x+1)) + \cdots\cdots + 1) + 1] \end{cases} \tag{9.13}$$

The analytic solution of $y = f_2(x)$ is

$$\begin{aligned} x &= [(Cd)_n]_{f_2}[ln(e^{(e^{\cdot^{\cdot^{(e^{y-f(\varphi_2)}-1)}}-\cdot-\cdot-\cdot-1)}} - 1)] \\ &= [(Cd)_n]_{f_2}[ln(ln[(ln[......(ln[y - f(\varphi_1) + 1] + 1) + ... + 1] + 1) + 1])] \\ &= (Cd_1)_{f_2}[ln(y - f(\varphi_1))] \end{aligned}$$

Proof Let $f_2 : A \in \Re \longrightarrow]f(\varphi_1) \quad +\infty)$ be bijective. Based on the previous proof, We will also define a function $Cd_{f_2} :]f(\varphi_1) \quad +\infty) \longrightarrow A \in \Re$ and check that $(Cd)_{1f_2}[ln(y - f(\varphi_1))]$ is the inverse of f_2. In addition we will also show that an infinite number of conditional function are related to f such that for the previous case

$$V(x)_n = ln[(ln[......(ln[x - f(\varphi_2) + 1] + 1) + ... + 1] + 1) + 1] > 0 \tag{9.14}$$

$$V(x)_m = e^{(e^{\cdot^{\cdot^{(e^{x-f(\varphi_1)}-1)}-\cdot-\cdot-\cdot-1)}}} - 1 > 0 \tag{9.15}$$

Example 9.2.2. Given $e^{-cx} = a_0(x - r_1)(x - r_2)$, if $n > \frac{e}{2}$, one of the turning point is

$$t = Cd_h(\sqrt{ln(2n) - 1}) + ln(2n) \tag{9.16}$$

[with $h(t) = e^t - t$], and

$$f(\varphi_1) = e^{[Cd_h(\sqrt{ln(2n)-1})+ln(2n)]} - n[Cd_h(\sqrt{ln(2n) - 1}) + ln(2n)]^2 \tag{9.17}$$

In the set $f(\varphi_2) < y < +\infty$, the transformation is

$$e^{Cd_{f_2}(t)} - n[Cd_{f_2}(t)]^2 = e^t + e^{[Cd_h(\sqrt{ln(2n)-1})+ln(2n)]} - n[Cd_h(\sqrt{ln(2n) - 1}) + ln(2n)]^2 \tag{9.18}$$

which yields the final and unique solution

$$x = \frac{-Cd_{f_1}[ln(y - e^{[Cd_h(\sqrt{ln(2n)-1})+ln(2n)]} - n[Cd_h(\sqrt{ln(2n) - 1}) + ln(2n)]^2)]}{c} + \frac{(r_1 + r_2)}{2a_0}$$

where

$$y = [\frac{-(r_1 + r_2)^2}{4a_0} + r_1 r_2]e^{[\frac{-(r_1+r_2)c}{2a_0}]}$$

9.3 Third branch of transformation

Theorem 9.3.1. *Let $f_3 : \Re \longrightarrow]f(\varphi_2) \quad f(\varphi_1)[$ be a third branch of f. f_3 is non bijective whose derivative $f_3^{'}$ has two turning points. f_3 is also associated to an infinite number of conditional functions, $g_n = [(Cd)_n]_f$ and $g_m = [(Cd)_m]_f : \Re \longrightarrow \Re$, n and $m \in 1, 2, 3 \cdots$. The composite map of f_3 and its main conditional function $[g_1 = [(Cd)_1]_f : \Re \longrightarrow \Re]$, $(n = 1)$, is the main or first transformation defined by*

$$f_3 \circ Cd_{f_3}[x] = x^3 + \frac{3}{2}[f(\varphi_2) - f(\varphi_1)]^{1/3}x + \frac{f(\varphi_2) + f(\varphi_1)}{2} \tag{9.19}$$

The three analytic solutions of $y = f_3(x)$ are

$$x_k = Cd_{f_3}[Vcos[\frac{1}{3}arcos([\frac{y}{2} - \frac{f(\varphi_2) + f(\varphi_1)}{4}]\sqrt{\frac{8}{[f(\varphi_1) - f(\varphi_2)]}}) + \frac{2k\pi}{3}] \tag{9.20}$$

where

$$V = 2\sqrt{\frac{1}{2}[f(\varphi_1) - f(\varphi_2)]^{1/3}} \text{ and } k \in \{0, 1, 2\}$$

Proof: Let $f_3 : A \in \Re \longrightarrow]f(\varphi_2) \quad f(\varphi_1)[$ be a given function. In this case $f^{'}$ admits two turning points φ_1, φ_2 then $\exists(x_1, x_2) \in \Re^2(x_1 < \varphi_1 or \varphi_2 < x_2)$ such that $f(x_1) = f(x_2) \Rightarrow \mathrm{x}_1 \neq x_2$. So f is a non injective function and then non bijective. By definition, f is related to a conditional function in a way that

$$f_3 \circ Cd_{f_3}(x) = U(x) \tag{9.21}$$

In addition U and f_3 have also by definition the same domain with two turning points each. Assuming that $U(x) = x^3 + px + q$ is a cubic function, the transformation is expressed by $f_3 \circ Cd_{f_3}[x] = x^3 + px + q$, the problem consist in determining p and q to obtain the inverse of f defined in $f(\varphi_2) < y < f(\varphi_1)$. According to the Cardan's method cited by [14], real solutions of the equation $x^3 + px + q = t(p < 0, -\infty \leq q \leq +\infty)$ are $x_k = 2\sqrt{\frac{-p}{3}}cos[\frac{1}{3}arcos(\frac{t-q}{2}\sqrt{\frac{27}{-p^3}}) + \frac{2k\pi}{3}]$. Then

$$f \circ Cd_h[2\sqrt{\frac{-p}{3}}cos[\frac{1}{3}arcos(\frac{t-q}{2}\sqrt{\frac{27}{-p^3}}) + \frac{2k\pi}{3}]] = x \tag{9.22}$$

The relation shows that $\frac{(t-q)}{2}\sqrt{\frac{27}{-p^3}} > -1$ and $\frac{(t-q)}{2}\sqrt{\frac{27}{-p^3}} < 1$. After expending and by definition, $t > q - \frac{2}{\sqrt{-27/p^3}} = f(\varphi_2)$ and $t < q + \frac{2}{\sqrt{-27/p^3}} = f(\varphi_1)$. Resolution of the system yields to

$$q = \frac{f(\varphi_2) + f(\varphi_1)}{2} \text{ and } p = \frac{3}{2}(f(\varphi_2) - f(\varphi_1))^{1/3} \tag{9.23}$$

The transformation is expressed

$$f \circ cd_{f_3}[x] = x^3 + \frac{3}{2}(f(\varphi_2) - f(\varphi_1))^{1/3}x + \frac{f(\varphi_2) + f(\varphi_1)}{2} \tag{9.24}$$

By setting $x^3 + \frac{3}{2}(f(\varphi_2) - f(\varphi_1))^{1/3}x + \frac{f(\varphi_2) + f(\varphi_1)}{2} = y$ yields to the three previous solutions

Example 9.3.2. Given $e^{-cx} = a_0(x - r_1)(x - r_2)$, the reduced form of equation is

$e^t - nt^2 = y(n, t)$ If $n > \frac{e}{2}$, y has two turning points $t = Cd_h(\pm\sqrt{ln(2n) - 1}) + ln(2n)$ (with $h(t) = e^t - t$),

$$\begin{cases} f(\varphi_2) = e^{[Cd_h(-\sqrt{ln(2n)-1})+ln(2n)]} - n[Cd_h(-\sqrt{ln(2n) - 1}) + ln(2n)]^2 \quad \text{and} \\ f(\varphi_1) = e^{[Cd_h(\sqrt{ln(2n)-1})+ln(2n)]} - n[Cd_h(\sqrt{ln(2n) - 1}) + ln(2n)]^2 \end{cases} \tag{9.25}$$

In the set $f(\varphi_2) < y < f(\varphi_1)$, $t = Cd_{f_3}(u)$ thus,

$$e^{Cd_{f_3}(u)} - n[Cd_{f_3}(u)]^2 = u^3 + \frac{3}{2}(f(\varphi_2) - f(\varphi_1))^{1/3}u + \frac{f(\varphi_2) + f(\varphi_1)}{2} = y \tag{9.26}$$

The three solutions of equations are

$$x_k = \frac{-Cd_{f_3}(u)}{c} + \frac{(r_1 + r_2)}{2a_0} \tag{9.27}$$

where $k = 0, 1, 2$

$$u = 2\sqrt{\frac{1}{2}[f(\varphi_1) - f(\varphi_2)]^{1/3}}cos[\frac{1}{3}arcos([\frac{y}{2} - \frac{f(\varphi_2) + f(\varphi_1)}{4}]\sqrt{\frac{8}{[f(\varphi_1) - f(\varphi_2)]}}) + \frac{2k\pi}{3} \tag{9.28}$$

Remark 9.3.3. When $y = f(x)$ with $(y, y) = (f(\varphi_1), f(\varphi_2))$, the equation has two solutions and the first one is (φ_1, φ_2) respectively. The second one is deduced from relation $(x - \varphi_1)[R(x)] = f(x) - f(\varphi_1)$ or $(x - \varphi_2)[P(x)] = f(x) - f(\varphi_2)$ respectively where $R(x)$ and $P(x)$ are obtained by dividing $f(x) - f(\varphi_1)$ or $f(x) - f(\varphi_2)$ by $(x - \varphi_1)$ and $(x - \varphi_2)$, respectively.

10 Transformation 6 of Non Bijective Function

Theorem 10.1. *Let $f : \Re \longrightarrow [f(\varphi_1) \quad +\infty)$ be a non bijective function whose derivative $f^{'}$ has three turning points $\varphi_1 < \varphi_2 < \varphi_3$ with a global minimum. f has three branches.*

(i) Let $f_1 : A \in \Re \longrightarrow [f_1(\varphi_1) \quad f_1(\varphi_2)]$ be a branch of f. f_1 is non bijective and associated to an infinite number of conditional functions, $g_n = [(Cd)_n]_f$ and $g_m = [(Cd)_m]_f : \Re \longrightarrow \Re$, n and $m \in 1, 2, 3 \cdots$. The composite map of f_1 and the two main conditional function $[g_{1,2} = [(Cd)_{1,2}]_f : \Re \longrightarrow \Re]$, $(n = 1)$, are the main or first transformations expressed by

$$f_1 \circ Cd_{f_{(1,2)}}(x) = \begin{cases} f_1(\varphi_1) + [\sqrt{f_1(\varphi_2) - f_1(\varphi_1)} - x^2]^2 \\ f_1(\varphi_2) - [\sqrt{f_1(\varphi_2) - f_1(\varphi_1)} - x^2]^2 \end{cases} \tag{10.1}$$

Two consecutive conditional functions also satisfied relation (4.2). Secondary transformations gotten from (10.1) and (4.2), are expressed like in a case of the transformation 1 and 4 by the following changing $\sqrt{f_1(\varphi_2) - f_1(\varphi_1)} - x^2 = t$

Solutions of $y = f(x)$ using the main transformation are

$$x_{1.2.3.4} = \begin{cases} Cd_{f_1}(\pm\sqrt{-\sqrt{y - f_1(\varphi_1)} + \sqrt{f_1(\varphi_2) - f_1(\varphi_1)}}) \\ Cd_{f_2}(\pm\sqrt{-\sqrt{f_1(\varphi_2) - y} + \sqrt{f_1(\varphi_2) - f_1(\varphi_1)}}) \end{cases} \tag{10.2}$$

(ii) Let $f_2 : A \in \Re \longrightarrow [f_2(\varphi_3) \quad f_2(\varphi_1)[$ be a branch of f as defined to (5.2). f_2 is non bijective and associated to an infinite number of conditional functions. The main transformation is

$$f_2 \circ Cd_{f_3}(x) = f_2(\varphi_1) - [f_2(\varphi_1) - f_2(\varphi_3)]e^{-x^2} \tag{10.3}$$

The analytic solutions of $y = f(x)$ from the main transformation are

$$x_{1.2} = Cd_{f_2}(\pm\sqrt{-[ln(\frac{f_2(\varphi_1) - y}{f_2(\varphi_1) - f_2(\varphi_3)})]})$$

(iii) Let $f_3 : A \in \Re \longrightarrow [f(\varphi) \quad +\infty)$ be a given non bijective function. f_3 is associated to an infinite number of conditional functions, $g_n = [(Cd)_n]_f$ and $g_m = [(Cd)_m]_f : \Re \longrightarrow \Re$, n and $m \in 1, 2, 3 \cdots$. The composite map of f_3 and its main conditional function $[g_1 = [(Cd)_1]_f : \Re \longrightarrow \Re]$, $(n = 1)$, is the main or first transformation expressed by

$$f_3 \circ Cd_{f_3}(x) = \frac{1}{x^2} + f_3(\varphi_2) \tag{10.4}$$

Two consecutive conditional functions satisfied the relation

$$Cd_{f_n}(\frac{1}{x}) = Cd_{f_{n+1}}(\frac{1}{\sqrt{ln(x^2 + 1)}}) \tag{10.5}$$

From (10.4) and (10.5), secondary transformations are

$$\left\{\begin{array}{l} f_3 \circ [(Cd)_n]_{f_3}(x) = [e^{[e^{\cdot^{\cdot^{(e^{x^{-2}}-1)}}}\_\cdot\_\cdot-1]} - 1] + f_3(\varphi_2) \\ f_3 \circ [(Cd)_n]_{f_3}(x) = ln[ln(\cdots\cdots(ln(x^{-2}+1)) + \cdots\cdots + 1) + 1] + f_3(\varphi_2) \end{array}\right. \tag{10.6}$$

In addition, analytic solutions of $y = f(x)$ are

$$\begin{aligned} x_{1.2} &= [cd_n]_{f_3}\left(\frac{1}{\pm\sqrt{e^{(e^{\cdot^{\cdot^{(e^{y-f(\varphi)}-1)}}}\_\cdot\_\cdot\_\cdot-1)} - 1}}\right) \\ &= [Cd_m]_{f_3}\left(\frac{1}{\pm\sqrt{ln[(ln[......(ln[y - f(\varphi) + 1] + 1) + ... + 1] + 1) + 1]}}\right) \\ &= [Cd_1]_{f_3}\left(\frac{1}{\pm\sqrt{y - f_3(\varphi_2)}}\right) \end{aligned}$$

Let $f : \Re \longrightarrow \Re$ be a non bijective function whose derivative $f^{'}$ has three turning points $\varphi_1 < \varphi_2 < \varphi_3$ with a global maximum. f has three branches.

(i) Let $f_1 : A \in \Re \longrightarrow [f_1(\varphi_2) \quad f_1(\varphi_1)]$ be a branch of f. f_1 is non bijective and associated to an infinite number of conditional functions, $g_n = [(Cd)_n]_f : \Re \longrightarrow \Re$, $n \in 1, 2, 3\cdots$. The composite map of f_1 and the two main conditional function $[g_{1,2} = [(Cd)_{1,2}]_f : \Re \longrightarrow \Re]$, $(n = 1)$, are the main or first transformations defined by

$$f \circ Cd_{f_{(1,2)}}(x) = \left\{\begin{array}{l} f(\varphi_2) + [\sqrt{f(\varphi_1) - f(\varphi_2)} - x^2]^2 \\ f(\varphi_1) - [\sqrt{f(\varphi_1) - f(\varphi_2)} - x^2]^2 \end{array}\right. \tag{10.7}$$

.

For each one, right and left Transformations of order n are obtained by substituting n times, $[u(x)]^2(u(x) = \sqrt{f(\varphi_1) - f(\varphi_2)} - x^2)$ from the main transformation by $e^{[u(x)]^2} - 1$ and $ln([u(x)]^2+1)$, respectively. In addition in order n, x^2 of $u(x)$ from the main transformation is substituted by $e^{x^2} - 1$ and $ln(x^2 + 1)$, respectively. Considering the first substitution, functions are closed to f by the equations

$$\left\{\begin{array}{l} f \circ [(Cd)_n]_f(x) = f(\varphi_2) + U \\ f \circ [(Cd)_n]_h(x) = f(\varphi_1) - U \end{array}\right. \tag{10.8}$$

where

$$U = [e^{[e^{\cdot^{\cdot^{(e^{[\sqrt{ln[(ln[..(ln[f(\varphi_1)-f(\varphi_2)+1]+1)+..+1]+1)+1]} - [e^{[e^{\cdot^{\cdot^{(e^{x^2}-1)}}}\_\cdot\_\cdot-1]}-1]]^2}-1)}}}\_\cdot\_\cdot-1]} - 1]$$

Solutions of $y = f(x)$ using the main transformation are

$$x_{1.2.3.4} = \left\{\begin{array}{l} Cd_{f_1}(\pm\sqrt{-\sqrt{f(\varphi_1) - y} + \sqrt{f(\varphi_1) - f(\varphi_2)}}) \\ Cd_{f_2}(\pm\sqrt{-\sqrt{y - f(\varphi_2)} + \sqrt{f(\varphi_1) - f(\varphi_2)}}) \end{array}\right. \tag{10.9}$$

(ii) *Let* $f_2 : A \in \Re \longrightarrow [f_2(\varphi_1) \quad f_2(\varphi_3)[$ *be a branch of* f *as defined to* (5.2). f_2 *is non bijective and associated to an infinite number of conditional functions. The main transformation is*

$$f \circ Cd_{f_3}(x) = f(\varphi_1) + [f(\varphi_3) - f(\varphi_1)]e^{-x^2} \tag{10.10}$$

The analytic solutions of $y = f(x)$ *are*

$$x_{1.2} = Cd_{f_3}(\pm\sqrt{ln[\frac{f(\varphi_3) - f(\varphi_1)}{y - f(\varphi_1)}]})$$

(iii) *Let* $f_3 : A \in \Re \longrightarrow] - \infty \quad f(\varphi_2)[$ *be a given non bijective function.* f_3 *is associated to an infinite number of conditional functions,* $g_n = [(Cd)_n]_f : \Re \longrightarrow \Re$, $n \in 1, 2, 3 \cdots$. *The composite map of* f_3 *and its main conditional function* $[g_1 = [(Cd)_1]_f : \Re \longrightarrow \Re]$, $(n = 1)$, *is the main or first transformation expressed by*

$$f \circ Cd_{f_4}(x) = f(\varphi_2) - \frac{1}{x^2} \tag{10.11}$$

Two consecutive conditional functions satisfied the relation of (10.5). *Secondary transformations deduced from* (10.11) *and* (10.5) *are*

$$\begin{cases} f \circ [(Cd)_n]_f(x) = f(\varphi_2) - [e^{[e^{\cdot^{\cdot^{(e^{x^{-2}}-1)}}}-\cdot-\cdot-1]} - 1] \\ f \circ [(Cd)_n]_f(x) = f(\varphi_2) - ln[ln(\cdots\cdots(ln(x^{-2} + 1)) + \cdots\cdots + 1) + 1] \end{cases} \tag{10.12}$$

The analytic solutions of $y = f(x)$ *are*

$$\begin{aligned} x_{1.2} &= [cd_n]_f(\frac{1}{\pm\sqrt{e^{(e^{\cdot^{\cdot^{(e^{f(\varphi_2)-y}-1)}}}-\cdot-\cdot-\cdot-1)} - 1}}) \\ &= [Cd_n]_f(\frac{1}{\pm\sqrt{ln[(ln[......(ln[f(\varphi_2) - y + 1] + 1) + ... + 1] + 1) + 1]}}) \\ &= Cd_{f_4}(\frac{1}{\pm\sqrt{f(\varphi_2) - y}}) \end{aligned}$$

Proof: Let $f_1 : A \in \Re \longrightarrow [f_1(\varphi_1) \quad f_1(\varphi_2)]$ be a branch of f. In this case $f_1^{'}$ admits three turning points φ_1, φ_2 and φ_3. Then $\exists(x_1, x_2) \in \Re^2$ $(x_1 < \varphi_1$ or φ_2 or $\varphi_3 < x_2)$ such that $f(x_1) = f(x_2) \Rightarrow$ $\mathrm{x}_1 \neq x_2$. So f is a non injective function and then non bijective. By definition, f_1 is related to a conditional function in a way that

$$f_1 \circ Cd_{f_1}(x) = U(x) \tag{10.13}$$

In that condition, f_1 has four reciprocal functions defined in the domain $[f_1(\varphi_1) \quad f_1(\varphi_2)]$ whose are both bijective because $\forall[(x_1, x_2, x_3, x_4) \in [f_1(\varphi_1) \quad f_1(\varphi_2)]^4]$, $\exists!(y_1, y_2, y_3, y_4)$ in their reciprocal set such that $y_1 = f_1(x_1)$, $y_2 = f_2(x_2)$, $y_3 = f_1(x_3)$ and $y_4 = f_1(x_4)$, respectively .

In addition U and f_1 by definition have the same domain with three turning points each. Assuming that $U(x) = x^4 + px^2 + q$ is a quartic function, the transformation is expressed by $f_1 \circ Cd_{f_1}[x] = x^4 + px^2 + q$. A changing of variable $x^4 + px^2 + q = \pm t$ with $p = 2k$ and $p = 2k_1$ leads to two cases.

$$x_1 = \pm\sqrt{-[\pm\sqrt{k^2 - q + t} + k]} \text{ and } x_2 = \pm\sqrt{-[\pm\sqrt{k_1^2 - q - t} + k_1]} \tag{10.14}$$

So (10.13) becomes

$$f_1 \circ Cd_{f_1}(\pm\sqrt{-[\pm\sqrt{k^2 - q + t} + k]}) = t \text{ and } f_1 \circ Cd_{f_1}(\pm\sqrt{-[\pm\sqrt{k_1^2 - q - t} + k_1]}) = t \tag{10.15}$$

So, to satisfy conditions $f(\varphi_1) \geq t \leq f(\varphi_2)$ of (10.15) we must have

$$k_1^2 - q - t \geq 0 \text{ and } [\sqrt{k_1^2 - q - t} + k] \leq 0 \tag{10.16}$$

In addition,

$$t + k^2 - q \geq 0 \text{ and } [\sqrt{t + k^2 - q} + k] \leq 0 \tag{10.17}$$

From these conditions, $t \leq k_1^2 - q = f_1(\varphi_2)$ with $q = -f_1(\varphi_1)$ and $t \geq -k^2 + q = f_1(\varphi_1)$ with $q = f_1(\varphi_2)$.

In the two cases,

$$k = k_1 = -\sqrt{f_1(\varphi_2) - f_1(\varphi_1)}$$

By taking $\sqrt{f_1(\varphi_2) - t} - \sqrt{f_1(\varphi_2) - f_1(\varphi_1)} = -x^2$ and $\sqrt{t - f_1(\varphi_1)} - \sqrt{f_1(\varphi_2) - f_1(\varphi_1)} = -x^2$, we obtain

$$\begin{cases} f_1 \circ g(x) = f_1(\varphi_2) - [\sqrt{f_1(\varphi_2) - f_1(\varphi_1)} - x^2]^2 \\ f_1 \circ h(x) = f_1(\varphi_1) + [\sqrt{f_1(\varphi_2) - f_1(\varphi_1)} - x^2]^2 \end{cases} \tag{10.18}$$

From (10.18), if $\sqrt{f(\varphi_2) - f(\varphi_1)} - t^2 = x$, we get transformations 1 and 4 and can apply the relation (4.2), (4.3) and (7.2) In the case where the curve is concave downwards,the same method is used to satisfy relation $f(\varphi_3) \leq x \leq f(\varphi_1)$

In the set $f(\varphi_2) < y < +\infty$, according to (2.1) and the fact that f has two branches, $Cd_f[U(x)]$ is the reciprocal function such that

$$f \circ Cd_f[U(x)] = x \text{ or } f \circ Cd_f[x] = U(x)^{-1} \tag{10.19}$$

then, conditions $Cd_f \in \Re^*$ and $f(\varphi_2) \leq Cd_f[U(x)] < +\infty$ are verified if

$$U(x)^{-1} = \frac{1}{x^2} + f(\varphi_2) \text{ and } u(x) = \frac{1}{\pm\sqrt{[x - f(\varphi_2)]}} \tag{10.20}$$

where $Cd_f[\frac{1}{\pm\sqrt{[x - f(\varphi_2)]}}]$ are the two branches of reciprocal function. Then

$$f \circ Cd_f[\frac{1}{\pm\sqrt{[x - f(\varphi_2)]}}] = x \text{ or } f \circ Cd_f[x] = \frac{1}{x^2} + f(\varphi_2) \tag{10.21}$$

is the main transformation. Many other infinite functions are related to f and proved as in the case of transformation 1

In the case where the curve has a global maximum,the same method is used to satisfy relation in the set $-\infty < y < f(\varphi_2)$

The particularity of that transformation is the using to solve any type of equation respecting our previous conditions such that quartic functions [12][15]

Example 10.2. *Let* $\frac{e^{2x}}{2} - 2e^x - \frac{x^3}{3} - x^2 = f(x) = y$. *The derivative is* $y'(x) = (e^x + x)(e^x - x - 2)$ *with* $(g(x), h(x)) = (e^x + x, e^x - x - 2)$. *The derivative has three turning points* $\varphi_1 = Cd_h(-1) = (Cd_1)_h(-\sqrt{ln(2)}) = -1.84168719 < \varphi_2 = Cd_g(0) = -0.56714329 < \varphi_3 = Cd_h(1) = (Cd_1)_h(\sqrt{ln(2)}) = 1.14619322$ *with a concave upwards curve.*

If $f[Cd_h(-1)] \leq y \leq f[Cd_g(0)]$, $f[Cd_h(-1)] \leq y \leq f[Cd_g(0)]$, $f[Cd_h(1)] \leq y < f[Cd_h(-1)]$ *and* $f[Cd_g(0)] < y < +\infty$, *the solutions of equations using mains transformations are*

$$x = \begin{cases} Cd_{f_1}(\pm\sqrt{-\sqrt{f[Cd_g(0)] - y} + \sqrt{f[Cd_g(0)] - f[Cd_h(-1)]}}) \\ Cd_{f_2}(\pm\sqrt{-\sqrt{y - f(Cd_h(-1))} + \sqrt{f(Cd_g(0)) - f(Cd_h(-1))}}) \\ Cd_{f_3}(\pm\sqrt{-[ln(\frac{f[Cd_h(-1)] - y}{f[Cd_h(-1)] - f[Cd_h(1)]})]}) \\ Cd_{f_4}(\frac{1}{\pm\sqrt{y - f[Cd_g(0)]}}) \end{cases} \tag{10.22}$$

respectively,

11 Numerical Evaluation of Conditional Function

Many methods of numerical evaluation have been developed. For each given function (f), an algebraic transformation is established. The conditional function (Cd_f) may be easily approximated using Newton's method [16]. It consists in defining a series $u_{(n+1)} = g(u_n)$ that converge to x_0 and where

$$x_0 \leq g(x) \leq x \text{ and } g(x) = x - \frac{f(x)}{f(x)'} \tag{11.1}$$

With successive approximations, some values of conditional function are resumed in the Table 1. For example

$$\begin{cases} X_1 = Cd_F(1) = ln[Cd_g(1)] = ln[-W_0(-e^{-2})] = 1.14619322\cdots \\ X_2 = Cd_F(-1) = ln[Cd_g(-1)] = ln[-W_{-1}(-e^{-2})] = -1.84168719\cdots \end{cases} \tag{11.2}$$

are solutions of equation $e^X - X = 2$ with $F(X) = e^X - X$ and $g(X) = X - lnX$. In addition,

$$\begin{cases} X_1 = Cd_g(1) = e^{Cd_F(1)} = -W_0(-e^{-2}) = 3.146193221\cdots \\ X_2 = Cd_g(-1) = e^{Cd_F(-1)} = -W_{-1}(-e^{-2}) = 0.158594339\cdots \end{cases} \tag{11.3}$$

are solutions of equation $X - lnX = 2$ with $F(X) = e^X - X$ and $g(X) = X - lnX$ (Table 1).

Table 1. Some remarkable and usual values of the conditional function

Reduced function	Values of Cd	Values of Cd
$f(x) = e^x + x$	$Cd_f(0) = -0.56714329$	$Cd_f(-2) = -2.120028239$
$f(x) = e^x + x$	$Cd_f(1) = 0$	$Cd_f(-1 + \mathrm{i}\pi) = \mathrm{i}\pi$
$f(x) = e^x + x$	$Cd_f(2) = 0.44285440$	$Cd_f(e^n + n) = n$
$f(x) = e^x + x$	$Cd_f(-1) = -1.278464543$	$Cd_f(n) \approx n\ (n \leq -20)$
$f(x) = e^x + x$	$Cd_f(-19) = -19.00000001$	$Cd_f(n) \approx ln(n)\ (n \geq e^{20})$
$g(x) = e^x - x$	$Cd_g(0) = 0$	$[Cd_1]_g(0) = 0$
$g(x) = e^x - x$	$Cd_g(1) = 1.14619322$	$e^{Cd_g(\pm i)} = Cd_g(\pm \mathrm{i})$
$g(x) = e^x - x$	$Cd_g(-1) = -1.84168719$	$Cd_g(\pm\sqrt{ln(0)}) = Cd_g(\pm \mathrm{i}) = Cd_h(-1)$
$g(x) = e^x - x$	$Cd_g(-2) = -4.993253432$	$e^{[Cd_1]_g(\pm i)} - [Cd_1]_f(\pm i) = \frac{1}{e}$
$h(x) = xe^x$	$Cd_h(0) =$ impossible in $\Re$	$Cd_h(-lna/a) = -lna\ (1/e \leq a \leq e)$
$h(x) = xe^x$	$Cd_h(e) = 1$	$Cd_h(1) = 0.56714329$
$h(x) = xe^x$	$Cd_h(-1/e) = -1$	$Cd_h(-1) = -0.31813 - 1.3372\mathrm{i}$
$h(x) = xe^x$	$Cd_h(-\pi/2) = (\pi/2)i$	$Cd'_h(0) =$ impossible in $\Re$
$k(x) = x - ln(x)$	$Cd_k(0) = 1$	$[Cd_1]_k(0) = 1$
$k(x) = x - ln(x)$	$Cd_k(1) = 3.146193221$	$[Cd_1]_k(1) = 4.138651946$
$k(x) = x - ln(x)$	$Cd_k(2) = 6.936847406$	$[Cd_1]_k(2) = 58.67007991$
$k(x) = x - ln(x)$	$Cd_k(-1) = 0.158594339$	$[Cd_1]_k(-1) = 0.070831586$
$k(x) = x - ln(x)$	$Cd_k(-2) = 0.006783381$	$[Cd_1]_k(\pm \mathrm{i}) - ln[[Cd_1]_k(\pm \mathrm{i})] = \frac{1}{e}$
$k(x) = x - ln(x)$	$Cd_k(\pm \mathrm{i}) = ln[Cd_k(\pm \mathrm{i})]$	$Cd_k(\pm \mathrm{i}) = [Cd_1]_k(\pm\sqrt{ln0})$
$v(x) = e^{x^2}$	$Cd_v(0) = 0$	$Cd_v(\pm i\sqrt{2}) = \pm\sqrt{\pi}\mathrm{i}$
$v(x) = e^{x^2}$	$Cd_v(\pm 1) = \pm\sqrt{ln(2)}$	$Cd_v(\pm \mathrm{i}) = \pm\sqrt{ln(0)}$
$v(x) = e^{x^2}$	$[Cd_1]_v(n) = n$	$Cd_v(\pm 0.794352761\mathrm{i}) = \pm \mathrm{i}$

12 Concluding Remarks

We noted that properties of conditional function and different related transformations are numerous and might be applied to dynamics of natural phenomena. Most of the functions usually used such that logarithm, exponential, Omega... can be expressed in terms of the conditional function. It is useful to implement the conditional function in a software. In addition, conditional function might be associated to any other function whose derivative admits more than three turning points. A transformation which is the composite map of f and its conditional function should be extended to the complex plane.

Competing Interests

Author has declared that no competing interests exist.

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