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A Numerical Method Based Boundary Integral Equation to Solve Multiphase Moving Boundary Problems

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Authors' contributions

This work was carried out in collaboration between all authors. Author SGA did the main idea, mathematical formulation and designation of the iterative scheme. Authors MSAD and MYY used the computer code designed by the author SGA. Authors MSAD and MYY carried all computations. Author SGA revised all computations. All authors read and approved the final manuscript.

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Abstract

The present paper deals with moving boundary problems by fixing the moving boundary by an assumption of the form $s(t)=2\varsigma\sqrt[m]{t}$, $m\geq 2$, m is assumed power and ς is an unknown. An iterative algorithm is then developed within the main algorithm to solve the phases that appear throughout the whole process with moving boundaries at each time step as phases with fixed boundaries. A two test problems are solved using the present method. The results due to the first test problem were compared with previous numerical results based on boundary integral formulation, while the results due to the second one was compared with available analytical solution. An overall good agreement is obtained for both two examples compared with the previous numerical and analytical results.

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1 Introduction

Phase change problem is a practical example of moving boundary problems and usually highly nonlinear due to the moving interface conditions [1-3]. Practical engineering problems are efficiently solved nowadays only by numerical methods, such as finite difference [4], finite element [5-7] and boundary elements [8-9]. Boundary integral method is very convenient to use for solution of Stefan problems in which, nodal points are located only on the boundaries and move together with the phase change interface, this means that, there is no need for any mesh adjustment [10-11]. The present paper deals with moving boundary problems by fixing the moving boundary by an assumption of the form $s(t) = 2\varsigma\sqrt[m]{t}$, $m \geq 2$, m is assumed power and ς is an unknown. An iterative algorithm is then developed within the main algorithm to solve the phases that appear throughout the whole process with moving boundaries at each time step as phases with fixed boundaries. A two test problems are solved using the present method. The results due to the first test problem were compared with previous numerical results based on boundary integral formulation, while the results due to the second one was compared with available analytical solution. An overall good agreement is obtained for both two examples compared with the previous numerical and analytical results.

2 Physical Background and Mathematical Formulation

2.1 Physical background

Melting and solidification are classical types of Stefan problems. Assume that we have a mould of finite length filled with liquid – in case of melting – and this mould is subjected to cooled air, then solidification starts appearing while the remaining still liquid and a moving interface between the two phases starts appearing and its position varies with time. Then the main unknown in this process is to trace the moving boundary with time. Once the moving boundary becomes known at each time step, then all other unknowns become easy to found. The problem underhand is classical melting or solidification Stefan's problem in which a new technique to trace the moving boundary is developed within main algorithm to solve such problems.

2.2 Mathematical formulation

A domain Ω consists of Ω_S -solid phase- and Ω_ℓ -liquid phase- representing the overall domain. The domain is bounded by a boundary Γ, while the two phases are separated by a moving interface Γ_m, see Fig. 1.

The mathematical formulation is as follows:

$$\alpha_i \nabla^2 u_i(\mathrm{x},t) = \frac{\partial u_i(\mathrm{x},t)}{\partial t}, \quad i = s, \ell \tag{1}$$

The subscripts $i = s, \ell$ refer to solid and liquid phases respectively. On the fixed boundary, two boundary conditions are prescribed:

$$u(\mathbf{x},t) = u_o \qquad \forall \mathbf{x} \in \Gamma_{f_1} \tag{2}$$

In case of ***solidification***, we have:

$$K_s\left(\frac{\partial u(\mathbf{x},t)}{\partial n}\right)=q(\mathbf{x},t),\ \forall \mathbf{x}\in\Gamma_{f_2} \tag{3}$$

Or in case of ***melting***, we have:

$$K_\ell\left(\frac{\partial u(\mathbf{x},t)}{\partial n}\right)=q(\mathbf{x},t),\ \forall \mathbf{x}\in\Gamma_{f_2} \tag{4}$$

In equations (2-4) $\Gamma=\Gamma_{f_1}\cup\Gamma_{f_2}$ represent the boundary of the domain Ω. The boundary condition (3) or (4) depends mainly on the type of the problem under consideration. On the moving boundary, two boundary conditions are prescribed:

$$u(\mathbf{x},t)=u_m \ \ \textbf{or}\ \ u_f,\ \ \forall \mathbf{x}\in\Gamma_m \ \ \ \textbf{or}\ \ \Gamma_f \tag{5}$$

$$\textbf{Input heat flux - output heat flux}=\pm\rho L V_n \tag{6}$$

$$u(\mathbf{x},0)=u_i \tag{7}$$

Based on the boundary integral formulation for fixed boundary [8], for any point the integral equation takes the following form:

$$u(\xi,\tau)=\int_0^\ell u(x,0)u^*(\xi,x;\tau,0)dx+k\int_0^\tau\int_0^\ell u^*(\xi,x;\tau,t)q(x,t)dxdt-k\int_0^\tau\int_0^\ell q^*(\xi,x;\tau,t)u(x,t)dxdt \tag{8}$$

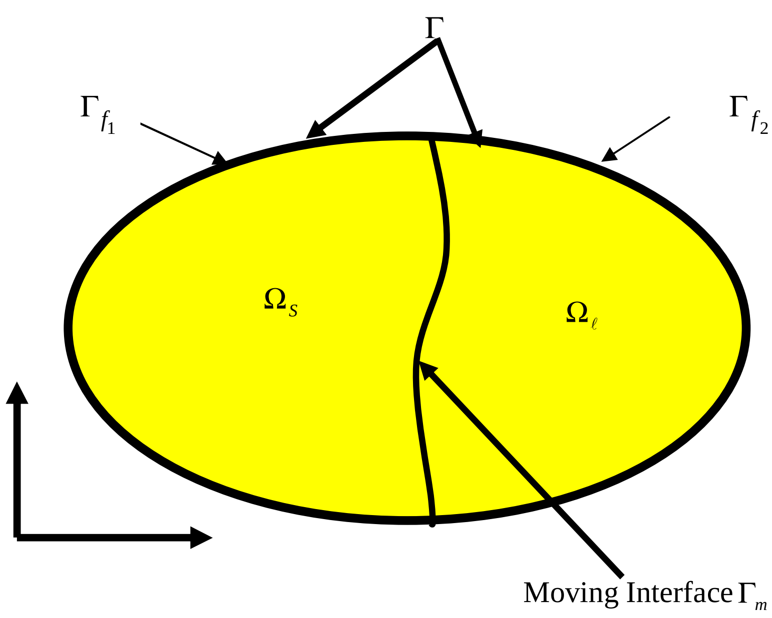

Fig. 1. Problem domain

3 Numerical Iterative Algorithm

In this section, the suggested iterative algorithm starting from determination the time at which phase change starts occurring up to the end of the process is shown in Fig. 2. As it appears, the first part of the developed algorithm is designed to determine the time at which phase change starts occurring within a prescribed error to ensure high accuracy of the computed results.

The algorithm in few steps:

1- Assume linear variation between the moving boundary and its speed, with $m = 2$ in the moving boundary equation.

2- Assume an initial position and subsequently initial speed.

3- Solve the two phases as fixed boundary problems with prescribed allowable error, then two possible outputs will occur, the first output is an achievement the prescribed error then move to next time step. The second output is a non-achievement of the prescribed error, then updating for both moving boundary position and its speed, then repeat again the process.

The details of the proposed algorithm are shown in the flow chart in Fig. 2.

4 Numerical Results

In the present paper, two different examples are solved, the first one is melting problem and the results due to the present method are compared with previous numerical results. The second problem is oxygen concentration and the results are compared with previous analytical results. The details for both examples are in the next subsections.

4.1 Example (1)

A solid medium initially at uniform temperature, $U_i = 300^o K$, the boundary $x = 0$ exposed to two different cases of input heat flux, constant, $Q(t) = 5\times10^6$ and linear, $Q(t) = 3\times10^4 t$ respectively. Fig. 3 shows the movement of solid-liquid due to constant heat flux, and the resulting ablation surface due to the same heat flux is shown in Fig. 4. The same results due to linear case are plotted on the same plot as shown in Figs. 5-6. From the above figures, it is found that the solid-liquid interface has the same behavior in both constant and linear cases of heat flux that is concave upward. In case of linear heat flux input this concavity becomes more apparent than the constant case. In the contrary, the ablated interface behaves concave downward but in linear case this concavity increases.

Also, It is found from the computation that the exponent m has a direct effect in the obtained results where the deviation from the previous moving code results starts increasing by increasing m but still acceptable up to $m = 5$. But this conclusion is not acceptable when tracing ablation interface, see Figs. 4 and 6, respectively.

4.2 Example (2)

Assume that the free surface of the solid $x = 0$, is exposed to a constant oxygen concentration C_o, and initial oxygen concentration is zero. Assume that oxygen concentration in oxidized and metallic layers is denoted by C_1 and C_2 respectively, the state equations describing this process, see [12]. The following numerical data are used in the computations:

$$D_1 = 0.274\mu m^2,\ D_2 = 0.166\mu m^2,\ C_o = 1.0,\ C_{cr} = 0.65 \text{ and } [C] = 0.25.$$

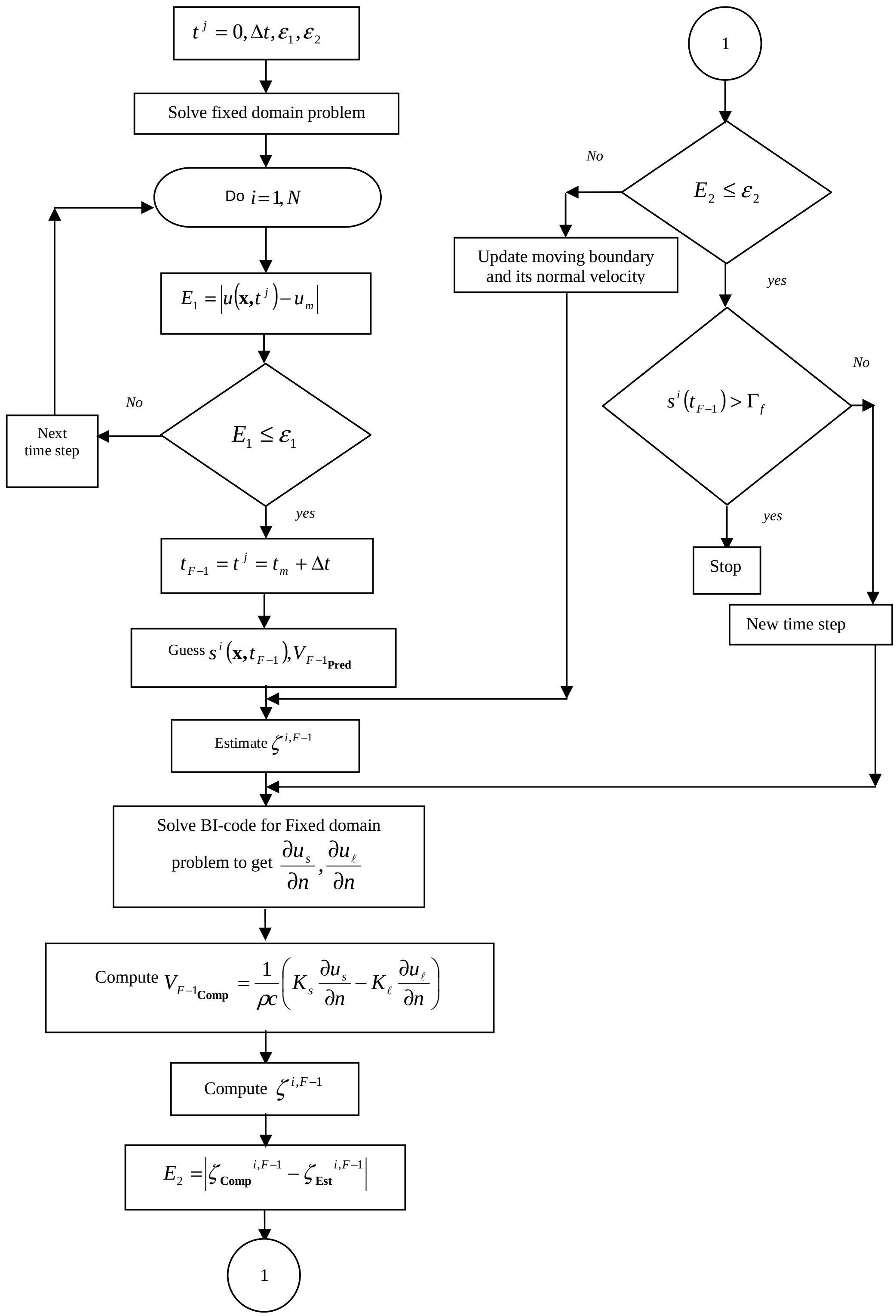

Fig. 2. Flow chart of the suggested algorithm

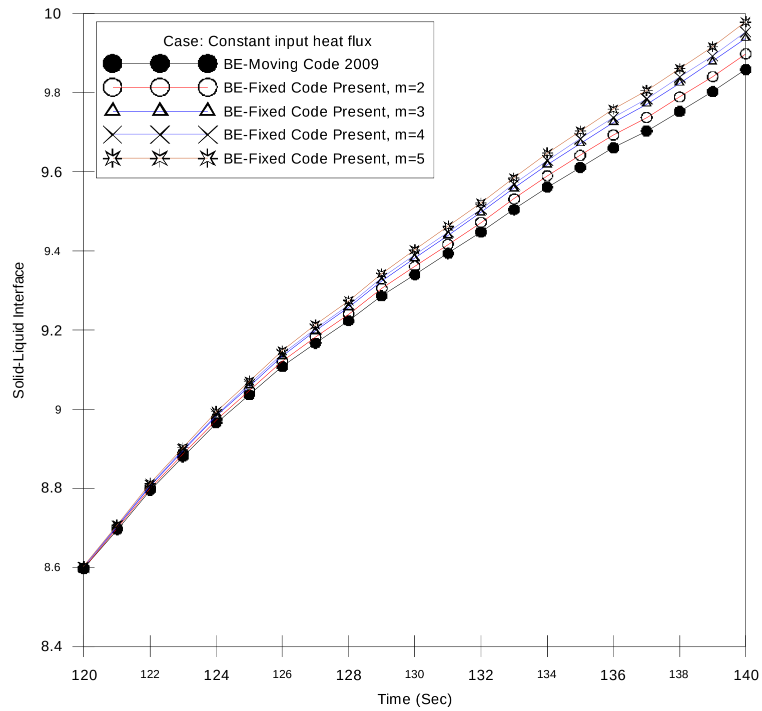

Fig. 3. Solid-Liquid due to constant input heat flux

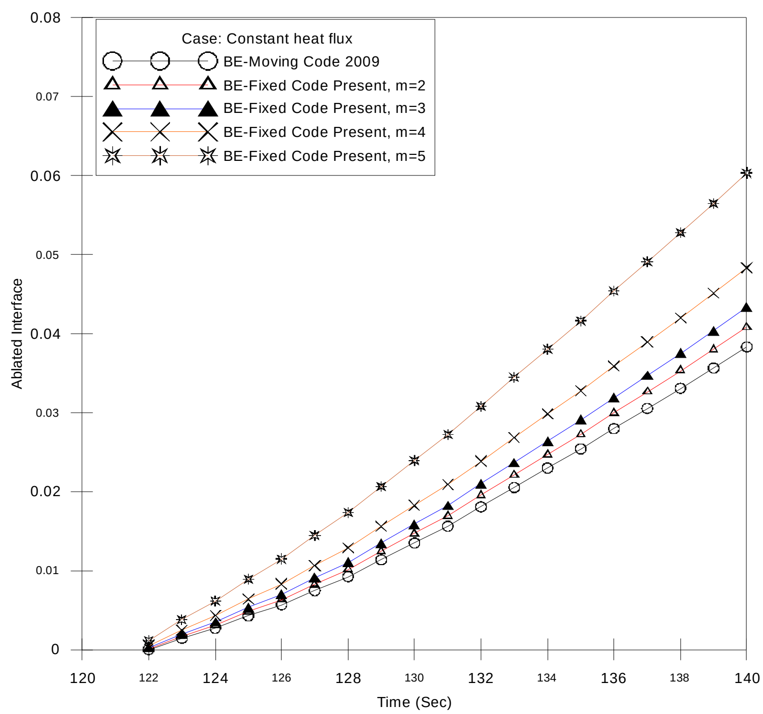

Fig. 4. Ablation surface due to constant heat flux

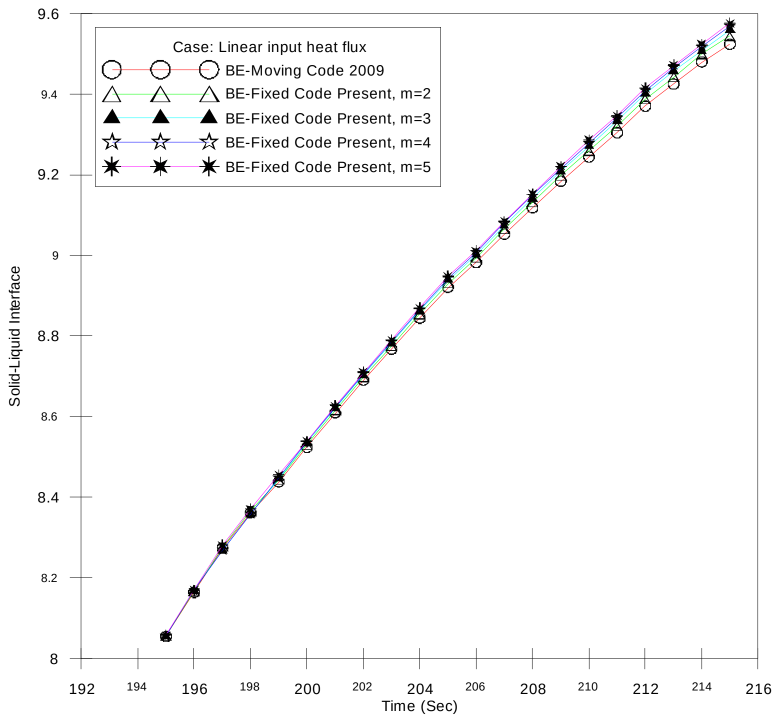

Fig. 5. Solid-liquid due to linear heat flux

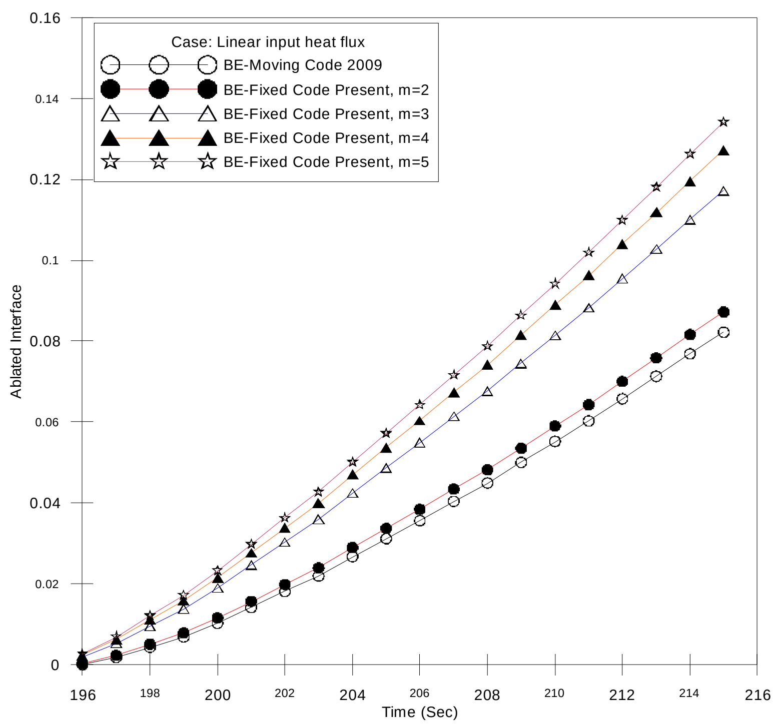

Fig. 6. Ablation surface due to linear heat flux

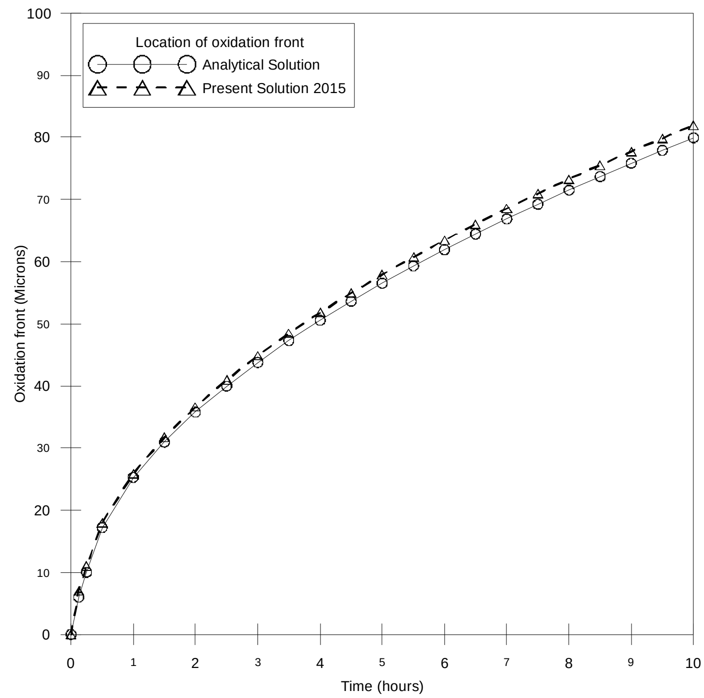

Fig. 7. Location of oxidation front

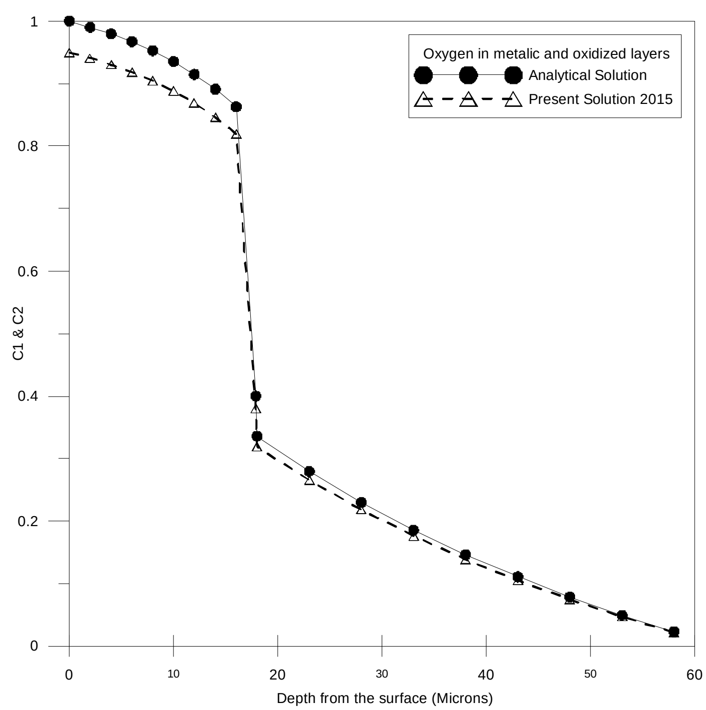

Fig. 8. Oxygen concentrations in both layers at time = 0.5 h

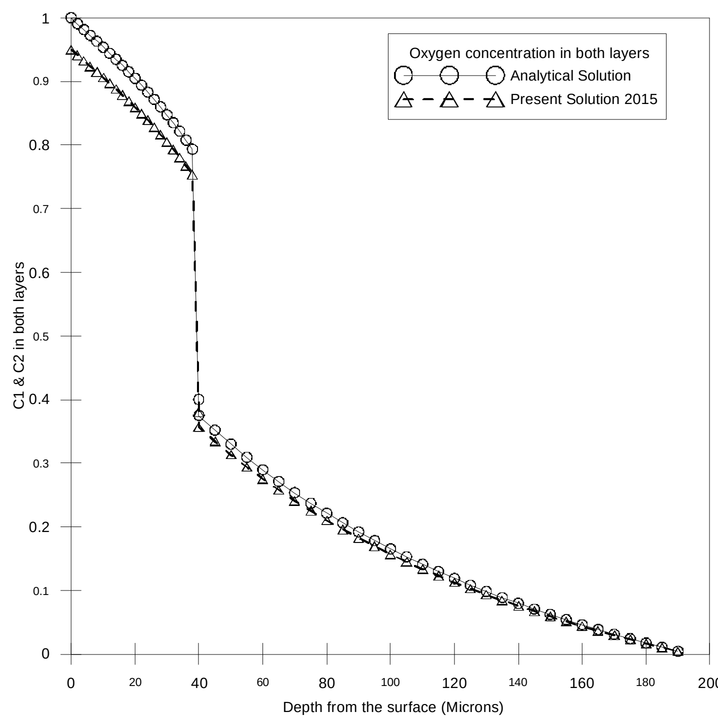

Fig. 9. Oxygen concentrations in both layers at time = 2.5 h

Since the exact analytical solution to the planar oxidation problem is available, a comparison between exact solution and the present method is made as shown in Fig. 7. It is clear that there is a good agreement between the two solutions with small acceptable error. Follow up the results of the present method, Figs. (8-9) show the oxygen concentration in both layers at different times $t = 0.5h$ and $t = 2.5h$. It is clear from these figures that the behavior is the same but the distance from the surface increases by growing up the time.

5 Conclusion

Numerical methods, techniques, and algorithms are long way and has no end as long as there exist scientific research, and the present paper is a trial in this long way. The main idea is to solve the moving boundary problem as a fixed boundary by a prescribed treatment of the moving boundary. An iterative scheme based on the boundary integral equation for fixed boundary was developed with prescribed allowable error. It is found from the computations the following:

1- The mathematics of the proposed technique is so simple compared with the boundary integral equation for domain with moving boundary.
2- Global number of iterations ranges from 20 to 25 iteration per time step.
3- Prescribed errors are small and acceptable for the practical applications.
4- The proposed hybrid technique can be modified to cover higher dimensional problems.
5- All moving boundary problems of like similarity moving interface can be easily solved using the proposed hybrid technique.

Competing Interests

Authors have declared that no competing interests exist.

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