British Journal of Mathematics & Computer Science
15(1): 1-15, 2016, Article no.BJMCS.24264
ISSN: 2231-0851

SCIENCEDOMAIN *international*
www.sciencedomain.org

Graph Compositions of Uniform Four q-fans

E. Mphako-Banda[1*], R. Kwashira[1] and B. Zinsou[1]

[1]*School of Mathematics, University of the Witwatersrand, P/Bag 3, WITS, 2050, South Africa.*

Authors' contributions

This work was carried out in collaboration between all authors. Author EMB designed the study. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/BJMCS/2016/24264
Editor(s):
(1) Morteza Seddighin, Indiana University East Richmond, USA.
Reviewers:
(1) Bibin K. Jose, S. D. College Alappuzha, Kerala, India.
(2) Sudev Naduvath, Vidya Academy of Science and Technology, Thrissur, India.
Complete Peer review History: http://sciencedomain.org/review-history/13528

Received: 11th January 2016
Accepted: 11th February 2016
Published: 3rd March 2016

Original Research Article

Abstract

Graph compositions play an important role in the generalization of both ordinary compositions of positive integers and partitions of finite sets. Graph compositions of certain classes of graphs, like trees, cycle graphs, wheels, etc have been found using generating functions and recurrence relations. In this paper, we use different combinatorial techniques, to count the number of graph compositions of uniform four q-fans.

Keywords: Graph composition; closed set; cycle matroid; fan.

2010 Mathematics Subject Classification: 05C30, 05A18, 05A05, 11B56.

*Corresponding author: E-mail: Eunice.Mphako-Banda@wits.ac.za;

1 Introduction

The notion of graph composition was introduced in [1]. In this paper various formulas, generating functions and recurrence relations for composition counting functions are given for several families of graphs. Graph compositions play an important role in the generalization of both ordinary compositions of positive integers and partitions of finite sets.

This work is extended further to bipartite graphs, some operations on graphs like unions of graphs and 2-sum of graphs, see [2], [3], [4]. In [5] the terminology of graph composition is explored further, but under a new term, comppartition. In [6] it is shown that the number of graph compositions is equal to the total number of flats of the cycle matroid.

For basic notation and terminology of graph theory, we refer the reader to [7]. The number of compositions for the following graphs: T_n, a tree on n vertices, P_n, a path with n vertices, S_n, a star graph on n vertices, K_n, a complete graph on n vertices, C_n, a cycle graph on n vertices, W_n, a wheel on n vertices, L_n, a ladder graph on $2n$ vertices and $K_{m,n}$, a complete bipartite graph is given in [1]. In particular, it is shown that $C(T_n) = C(P_n) = C(S_n) = 2^{n-1}$, where $C(G)$ is the number of graph compositions of the graph G.

There are several equivalent ways of defining a matroid, we refer the reader to Oxley [8]. Let G be a graph with $\omega(G)$ connected components and let E be the set of edges in G. Define a rank function r such that if $X \subseteq E$ then $r(X) = |V(G[X])| - \omega(G[X])$. It it easy to show that r is a rank function of a matroid on a set E. This matroid is called the cycle matroid of G and is denoted by $M(G)$. The closure operator is another function associated with a matroid. Let X be a rank r subset of E, then the closure of X denoted by $\mathrm{cl(X)}$ is the largest rank r subset of E containing X. In particular, $\mathrm{cl(X) = \{x \in E : r(X \cup x) = r(X)\}}$. A flat of a matroid $M(E)$ is a set $A \subseteq E$ for which $\mathrm{cl(A) = A}$. A flat of rank zero is called an empty flat.

The relationship between graph compositions and flats of cycle matroids was established in the following theorem, for details we refer to [6]

Theorem 1.1. *Let G be a labelled graph with vertex set $V(G)$ and edge set $E(G)$. Let $\mathcal{C}_o(G)$ be the set of all distinct compositions of G such that $C(G) = |\mathcal{C}_o(G)|$ and let $\mathcal{F}(M(G))$ be the set of all distinct flats of $M(G)$. Then $C(G) = |\mathcal{F}(M(G))|$.*

As a consequence of Theorem 1.1 we get the following result in Corollary 1.2, which is the main tool of this work.

Corollary 1.2. *Let G be a labelled graph with vertex set $V(G)$ and edge set $E(G)$. Then*

$$C(G) = \sum_{k=0}^{|E(G)|} |\delta_k|$$

where $|\delta_k|$ is the number of compositions of G of size k.

Theorem 1.1 opened a new way of looking at the methods of counting the number of graph compositions. One of the combinatorial techniques explored, is the principle of inclusion and exclusion, see [9].

This paper is a continuation of exploring some combinatorial techniques to count the number of graph compositions. We define a certain class of graphs and count the number of graph compositions for this class of graphs.

2 Fans

In this section we define an n-fan, and we extend this definition to a new class of graphs and give some of its characteristics. Recall that P_n is a tree which is a path on n vertices.

We define a *n-fan* to be the vertex join of a path on n vertices. We shall call an edge of an n-fan which is not on the path, P_n, a *join edge*. We define a uniform n_q-fan to be an n-fan such that each join edge is subdivided into q edges. We shall denote a uniform n_q-fan by F_{n_q}. The following Proposition follows from the definition of a uniform n_q-fan.

Proposition 2.1. *Let F_{n_q} denote a uniform n_q-fan. Then*

1. $|V(F_{n_q})| = nq + 1$
2. $|E(F_{n_q})| = n(q + 1) - 1$.

Proposition 2.2. *Let G be a uniform n_q-fan. Then there are subgraphs of G isomorphic to C_{2q+1}, C_{2q+2}, $\cdots$, C_{2q+n-1}.*

3 Graph Compositions of a Uniform 4_q-fan

In this section we give the number of graph compositions of a uniform 4_q-fan. It is shown in [6, Theorem 1.1] that the number of graph compositions of a graph G is equal to the number of closed sets of G, where closed sets are the flats of the cycle matroid of G. Thus we only have to count the number of closed sets of different sizes of a uniform 4_q-fan.

Let G be a graph on n vertices with c connected components. The rank of G, denoted $r(G)$, is defined to be $r(G) = n - c$. A closed set X of size k, is the largest rank-r subgraph of G containing X.

We denote the set of all closed sets of size k by δ_k. Thus the number of all closed sets of size k is represented by $|\delta_k|$. For a uniform 4_q-fan, different values of k give different formulae where $0 \leq k \leq 4q + 3$. The following trivial Lemma 3.1 is the implication of the result stated in [1].

Lemma 3.1. *Let P_n be the path of order n. Then any subset X of P_n is closed in P_n.*

The following Lemma 3.2 is trivial but plays a very important role in the proofs of the other lemmas and hence the foundation for the proof of the main theorem.

Lemma 3.2. *Let C_n be the cycle graph of order n and $X \subseteq C_n$. X is a non-closed subset in C_n iff $|E(X)| = n - 1$.*

Proof. Without loss of generality, let $V(C_n) = \{v_1, v_2, \cdots, v_{n-1}, v_n\}$ and
$E(C_n) = \{\{v_1, v_2\}, \{v_2, v_3\} \cdots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\}$.

Assume X is a non-closed subset in C_n. Then there exists, a larger subset of C_n, say Y containing X with $r(X) = r(Y)$. It is clear that all proper subsets of C_n are paths or union of paths contained in a certain path P_n. Thus by implication of Lemma 3.1, there is no subest $Y \cong P_t$ containing X with $r(X) = r(Y)$ except when $X \equiv Y$. Thus the only subset which can contain X is C_n and the only subsets of C_n with the same rank as C_n are the subsets $X \cong P_n$. i.e we get $r(X) = r(P_n) = n - 1 = r(C_n)$. Then $|E(X)| = E(P_n)| = n - 1$.

Assume $|E(X)| = n - 1$, $E(X) \subset E(C_n)$, say $E(X) = \{\{v_1, v_2\}, \{v_2, v_3\} \cdots, \{v_{n-1}, v_n\}\}$. Then X has one component and n vertices. Therefore the rank, $r(X) = r(C_n) = n - 1$, hence by definition X in a non-closed set because X has rank $n - 1$ but $C_n = X \cup \{a_n\}$ is larger and contains X and has also rank $n - 1$. $\square$

Lemma 3.3. *Let G be any graph on n vertices and let a cycle C_m be a subgraph of G where $m \leq n$. Then any subgraph $X \subset G$ containing $m-1$ edges only of C_m and other edges not in C_m is a non-closed subset.*

Proof. Let $E(G) = \{e_1, e_2, \cdots, e_{m-1}, e_m, a_1, a_2, \cdots, a_t\}$ such that $n > m$. Let $E(C_m) = \{e_1, e_2, \cdots, e_{m-1}, e_m\} \subset E(G)$ and let

$E(X) = \{e_1, e_2, \cdots, e_{m-1}, a_1, a_2, \cdots, a_l\} \subset E(G)$ where $l \leq t$. Then it follows that $r(X) = r(X \cup \{e_m\})$, thus X is not the largest rank-r subgraph containing X. Therefore X is a non-closed subset. $\square$

Corollary 3.4. *Let G be a graph such that $C_n \subset G$ and is the smallest cycle in G. Then any subset X such that $|E(X)| \leq n-2$ is closed in G.*

Proposition 3.1. *Let G be a uniform 4_q-fan. Then G has exactly*

(1) *3 subgraphs isomorphic to C_{2q+1},*

(2) *2 subgraphs isomorphic to C_{2q+2},*

(3) *1 subgraph isomorphic to C_{2q+3}.*

Unless otherwise stated, in the remainder of the paper, we use the diagram in Fig. 1 and the notation introduced in Notation 3.5.

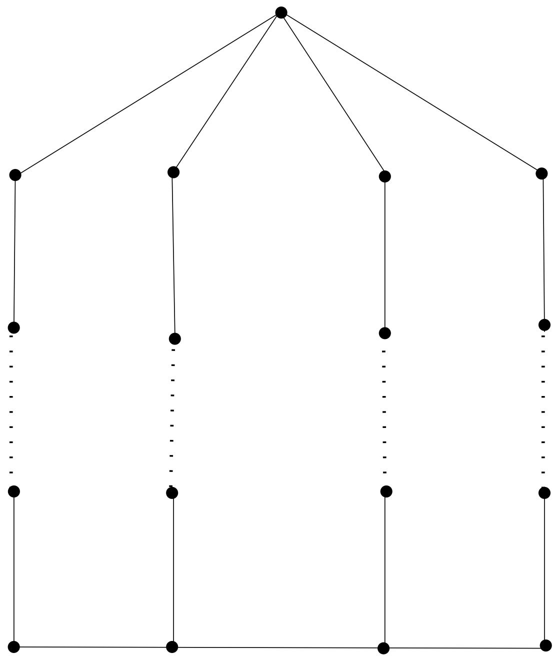

Fig. 1. A general 4_q-fan

Notation 3.5. *Consider a general F_{4_q}, a general 4_q-fan as shown in Fig. 1. Denote the three subgraphs isomorphic to C_{2q+1} of the graph F_{4_q}, as*

$$\begin{aligned} C_{t_1} =& \{a_{11}, a_{12}, \cdots, a_{1q}, b, a_{21}, a_{22}, \cdots, a_{2q}\}, \\ C_{t_2} =& \{a_{21}, a_{22}, \cdots, a_{2q}, c, a_{31}, a_{32}, \cdots, a_{3q}\}, \\ C_{t_3} =& \{a_{31}, a_{32}, \cdots, a_{3q}, d, a_{41}, a_{42}, \cdots, a_{4q}\}. \end{aligned}$$

Denote the two subgraphs isomorphic to C_{2q+2} of the graph F_{4_q}, as

$$\begin{aligned} C_{t_4} =& \{a_{11}, a_{12}, \cdots, a_{1q}, b, c, a_{31}, a_{32}, \cdots, a_{3q}\} \\ C_{t_5} =& \{a_{21}, a_{22}, \cdots, a_{2q}, c, d, a_{41}, a_{42}, \cdots, a_{4q}\}. \end{aligned}$$

Denote the one subgraph isomorphic to C_{2q+3} of the graph F_{4_q}, as

$$C_{t_6} = \{a_{11}, a_{12}, \cdots, a_{1q}, b, c, d, a_{41}, a_{42}, \cdots, a_{4q}\}.$$

We are now in a position to count the number of closed sets of a uniform 4_q-fan, F_{4_q}. Closed sets of different sizes k have different formulas, hence we give the formulas for each $k \in \{0, 1, 2, \cdots, 4q+3\}$. By applying Corollary 3.4 and Proposition 3.1, we get Proposition 3.2.

Proposition 3.2. *Let F_{4_q} be a uniform 4_q-fan with $q \geq 1$. Let δ_k represent the set of all closed sets of size k of F_{4_q}. Then for $0 \leq k \leq 2q-1$, we have*

$$|\delta_k| = \binom{4q+3}{k}.$$

Corollary 3.6. *Let F_{4_q} be a uniform 4_q-fan with $q \geq 1$. Let δ_k represent the set of all closed sets of size k of F_{4_q}. Then for $0 \leq k \leq 2q-1$*

$$\sum_{k=0}^{2q-1} |\delta_k| = \sum_{k=0}^{2q-1} \binom{4q+3}{k}.$$

Proposition 3.3. *Let F_{4_q} be a uniform 4_q-fan with $q \geq 1$. Let δ_k represent the set of all closed sets of size k of F_{4_q}. Then for $k = 2q$*

$$|\delta_k| = \binom{4q+3}{2q} - 3\binom{2q+1}{2q}\binom{2q+2}{k-2q}.$$

Proof. We have $\binom{4q+3}{2q}$ subsets of size $2q$ of F_{4_q}. But out of these subsets, some subsets are closed and some are non-closed subsets. Hence we remove the non-closed subsets of size $2q$. By Lemma 3.1 we have 3 subgraphs isomorphic to C_{2q+1}, 2 subgraphs isomorphic to C_{2q+2} and one subgraph isomorphic to C_{2q+3}. The only cycle which will affect the closed sets of size $2q$ is C_{2q+1}, since by Lemma 3.2 any $2q$ element subset of $C_{2q+1} \subset F_{4_q}$ is non-closed. Thus we have $3\binom{2q+1}{2q}$ non-closed subsets to remove with respect to the 3 subgraphs isomorphic to C_{2q+1}. □

To ease the computations at each stage, we state the following Lemma.

Proposition 3.4. *Let F_{4_q} be a uniform 4_q-fan with $q \geq 1$. Let δ_k represent the set of all closed sets of size $k \geq 2q+1$ of F_{4_q}. Then*

$$\begin{aligned} |\delta_k| \quad =& \quad \binom{4q+3}{k} - 3\binom{2q+1}{2q}\binom{2q+2}{k-2q} - 2\binom{2q+2}{2q+1}\binom{2q+1}{k-(2q+1)} \\ -& \quad \binom{2q+3}{2q+2}\binom{2q}{k-(2q+2)} + H_k(q). \end{aligned}$$

where $H_k(q)$ counts the number of repeats of non-closed sets which have been removed more that once.

Proof. We have $\binom{4q+3}{k}$ subsets of size k of F_{4_q}. But out of these subsets, some subsets are closed and some are non-closed subsets. Hence we remove the non-closed subsets of size k from the total. By Lemma 3.1 we have 3 subgraphs isomorphic to C_{2q+1}, 2 subgraphs isomorphic to C_{2q+2} and one subgraph isomorphic to C_{2q+3}. We start by removing the non-closed subsets with respect to cycle C_{2q+1}. By Lemma 3.3 any subset X which contains exactly $2q$ elements of C_{2q+1} is non-closed. To form a non-closed subset X, take a $2q$ element subset of C_{2q+1} and choose the other $k-2q$ elements from the remaining elements of F_{4_q} which are not in C_{2q+1}. Thus with respect to the 3 cycles, we have $3\binom{2q+1}{2q}\binom{2q+2}{k-2q}$ non-closed subsets to remove from the total. Similarly, with respect to the 2 cycles isomorphic to C_{2q+2} and 1 cycle isomorphic to C_{2q+3}, we have $2\binom{2q+2}{2q+1}\binom{2q+1}{k-2q-1}$ and $\binom{2q+3}{2q+2}\binom{2q}{k-2q-2}$ non-closed subsets respectively. □

If we denote the following numbers as

$$\begin{aligned}
\alpha &= \binom{4q+3}{k}, \\
\beta_1 &= 3\binom{2q+1}{2q}\binom{2q+2}{k-2q} = (6q+3)\binom{2q+2}{k-2q}, \\
\beta_2 &= 2\binom{2q+2}{2q+1}\binom{2q+1}{k-(2q+1)} = (4q+4)\binom{2q+1}{k-(2q+1)}, \\
\beta_3 &= \binom{2q+3}{2q+2}\binom{2q}{k-(2q+2)} = (2q+3)\binom{2q}{k-(2q+2)}.
\end{aligned}$$

then Proposition 3.4 simplifies to Corollary 3.7.

Corollary 3.7. *Let F_{4_q} be a uniform 4_q-fan with $q \geq 1$. Let δ_k represent the set of all closed sets of size $k \geq 2q+1$ of F_{4_q}. Then*

$$|\delta_k| \quad = \quad \alpha - \beta_1 - \beta_2 - \beta_3 + H_k(q).$$

Thus from this point, we only need to find the number $H_k(q)$ for different values of $k \geq 2q+1$.

Proposition 3.5. *Let F_{4_q} be a uniform 4_q-fan with $q \geq 1$. Let δ_k represent the set of all closed sets of size k of F_{4_q}, where $|\delta_k| = \alpha - \beta_1 - \beta_2 - \beta_3 + H_k(q)$. Then for $2q+1 \leq k \leq 3q-1$*

$$H_k(q) = 0.$$

Proof. We need to show that the non-closed sets removed with respect to C_{t_1}, C_{t_2} and C_{t_3} in the terms β_1, β_2 and β_3 are distinct. Assume that there is a non-closed subset X which has been removed twice in the term β_1. Without loss of generality, then X has $2q$ edges in C_{t_1} and $2q$ edges in C_{t_2} or C_{t_3}. This is only possible if $C_{t_1} \cap C_{t_2}$ or $C_{t_1} \cap C_{t_3}$ are non-empty, since the number of edges is less than or equal to $3q-1 < 4q$. There is no non-closed subset which has been removed with respect to C_{t_1} and C_{t_3} in the term β_1, since $C_{t_1} \cap C_{t_3} = \emptyset$. By Notation 3.5 we know that $|E(C_{t_1}) \cap E(C_{t_2})| = q$. Assume that X have $2q$ edges in C_{t_1} such that q of these edges are in $C_{t_1} \cap C_{t_2}$, then we need only $1 \leq j \leq q-1$ edges in C_{t_2} to form a non-closed set of size k. Thus X can only contain $q+q-1$ edges at most of C_{t_2} which is a contradiction.

Assume that there is a non-closed subset X which has been removed twice in the term β_2, without loss of generality, then X has $2q+1$ edges in C_{t_4} and $2q+1$ edges in C_{t_5}. Thus if X has $2q+1$ edges in C_{t_4} then only $0 \le j \le q-2$ can come from C_{t_5}. But $|E(C_{t_4}) \cap E(C_{t_5})| = 1$, hence X can have at most $q-1$ from C_{t_5}, which is a contradiction.

Assume that there is a non-closed subset X which has been removed in the terms β_1 and β_2, without loss of generality, then X has $2q$ edges in C_{t_1} and $2q+1$ edges in C_{t_4} or C_{t_5}. It is clear that this is only possible if $C_{t_1} \cap C_{t_4}$ or $C_{t_1} \cap C_{t_5}$ are non-empty. By Notation 3.5 $|E(C_{t_1}) \cap E(C_{t_4})| = q+1$ and $|E(C_{t_1}) \cap E(C_{t_5})| = q$. Assume that X has $2q$ edges in C_{t_1} such that $q+1$ of these edges are in $C_{t_1} \cap C_{t_4}$, then we need only $1 \le j \le q-1$ edges in C_{t_4} to form a non-closed set of size k. Thus X can only contain $q+1+q-1 = 2q$ edges at most of C_{t_4}. Similarly, X can only contain $q+q-1 = 2q-1$ edges at most of C_{t_5} which is a contradiction.

Similarly, if we assume that there is a non-closed subset X which has been removed in the terms β_1 and β_3 or in the terms β_2 and β_3, we get contradictions. □

Corollary 3.8. *Let F_{4_q} be a uniform 4_q-fan with $q \ge 1$. Let δ_k represent the set of all closed sets of size k of F_{4_q}. Then for $2q+1 \le k \le 3q-1$*

$$\begin{aligned}\sum_{k=2q+1}^{3q-1} |\delta_k| &= \sum_{k=2q+1}^{3q-1} \left[\binom{4q+3}{k} - (6q+3)\binom{2q+2}{k-2q} \right] \\ &+ \sum_{k=2q+1}^{3q-1} \left[-(4q+4)\binom{2q+1}{k-(2q+1)} - (2q+3)\binom{2q}{k-(2q+2)} \right].\end{aligned}$$

We need the following lemma for the proof of Proposition 3.6.

Lemma 3.9. *Let $X \subset F_{4_q}$ such that $|E(X)| = 3q$ and let $|E(X) \cap E(C_{t_6})| = 2q+2$. (i.e all the edges of C_{t_6}, except one, are contained in $E(X)$.) Then*

(i) *$E(X)$ can not contain any $2q$ edges of C_{t_1} or C_{t_3}.*

(ii) *$E(X)$ can not contain any $2q+1$ edges of C_{t_4} or C_{t_5}.*

Proof. 1. Let $X \cap C_{t_6} = Y$. Thus $|E(Y)| = 2q+2$. Let $G_{16} = C_{t_1} \cap C_{t_6}$ and $G_{36} = C_{t_3} \cap C_{t_6}$. Then it is clear that $|E(G_{16})| = q+1 = |E(G_{36})|$, $G_{16} \cap G_{36} = \emptyset$ and $C_{t_6} = G_{16} \cup G_{36} \cup \{c\}$. But $X = Y \cup X_p$ where $X_p \subset F_{4_q} - C_{t_6}$, thus $C_{t_6} \cap X_p = \emptyset$. Hence, $|E(X)| = |E(Y)| + |E(X_p)| = 3q$ and $|E(X_p)| = q-2$.

Now we find the maximum number of elements of C_{t_1} contained in X. Assume that $G_{16} \subset X$ and that all the $q-2$ elements of X_p are taken from $C_{t_1} - G_{16}$. Then the maximum number of elements in X from C_{t_1} is $|E(G_{16})| + |E(X_p)| = q+1+q-2 = 2q-1$.

2. Similar to proof of part (1).

□

Proposition 3.6. *Let F_{4_q} be a uniform 4_q-fan with $q \ge 1$. Let δ_k represent the set of all closed sets of size k of F_{4_q}, where $|\delta_k| = \alpha - \beta_1 - \beta_2 - \beta_3 + H_k(q)$. Then for $k = 3q$*

$$H_k(q) = 2\binom{q+1}{q}\binom{q+1}{q} + 4\binom{q}{q-1}\binom{q+1}{q}.$$

Proof. Some subsets of these subsets have been deducted more than once in the terms β_1, β_2 and β_3. We show how to get each term of $H_k(q)$ separately.

1. We consider non-closed subsets of size $3q$ which have been removed twice in the term β_1. Consider two intersecting cycles of size $2q+1$, say cycle C_{t_1} and cycle C_{t_2}. Then $C_{t_1} \cap C_{t_2} = \{a_{21}, a_{22}, \cdots, a_{2q}\}$, and we denote it as G_{12}. We denote any q element subset in $C_{t_1} - G_{12}$ as G_1 and any q element subset in $C_{t_2} - G_{12}$ as G_2. Hence any $3q$ element subset of the form $G_{12} \cup G_1 \cup G_2$ has been removed twice, hence we need to add it back once. But $|E(C_{t_1}) - E(G_{12})| = 2q+1-q = q+1$, thus we have $\binom{q+1}{q}$ subsets with q elements, G_1. Similarly we have $|E(C_{t_2}) - E(G_{12})| = 2q+1-q = q+1$, and $\binom{q+1}{q}$ subsets with q elements, G_2. We do similar counting for the the other intersecting cycles C_{t_2} and C_{t_3}. Hence we add back $2\binom{q+1}{q}\binom{q+1}{q}$.

2. We consider non-closed subsets of size $3q$ which have been removed in the terms β_1 and β_2. It is clear that $|E(C_{t_4}) \cap E(C_{t_1})| = |\{a_{11}, a_{12}, \cdots, a_{1q}, b\}| = q+1$. Take a $3q$ element subset where $2q+1$ elements are in C_{t_4} such that $q+1$ elements of these are in $C_{t_4} \cap C_{t_1}$, the other q elements are in $C_{t_4} - C_{t_1}$ and the remaining $q-1$ elements in G_{12}. Thus we have taken $2q$ elements from C_{t_1}, since by definition $G_{12} \subset C_{t_1}$. Then this is equivalent to a $3q$ element non-closed subset removed in the term β_1 in which $2q$ elements were taken from C_{t_1} and other q elements outside C_{t_1}, specifically in $C_{t_4} - C_{t_1}$, thus we need to add it back. There are $\binom{q}{q-1}$ ways of choosing the $q-1$ elements of G_{12} and $\binom{q+1}{q}$ ways of choosing the elements $C_{t_4} - C_{t_1}$. But we can swap C_{t_1} and C_{t_2} forming a $3q$ element non-closed subset in which $2q$ elements were taken in C_{t_2} and other q elements outside C_{t_2} specifically in $C_{t_4} - C_{t_2}$. Thus we have $2\binom{q+1}{q}\binom{q}{q-1}$ subsets with respect to C_{t_4} and similarly with respect to C_{t_5}. Hence we add back $4\binom{q+1}{q}\binom{q}{q-1}$ to the total.

3. Applying Lemma 3.3, any subgraph of size $3q$ which contains exactly $2q+2$ edges of cycle C_{t_6} is non-closed. Now we count these non-closed sets. Without loss of generality consider the $3q$ element subsets with exactly $2q+2$ elements in C_{t_6}. It is clear that there are $\binom{2q+3}{2q+2}\binom{2q}{q-2}$ subsets of size $3q$ of this form. By applying Lemma 3.9, it is clear that there are no subsets of size $3q$ of this form which have already been removed in the term β_1 or β_2.

□

Corollary 3.10. *Let F_{4_q} be a uniform 4_q-fan with $q \geq 1$. Let δ_k represent the set of all closed sets of size k of F_{4_q}. Then for $k = 3q$*

$$
\begin{aligned}
|\delta_k| &= \binom{4q+3}{k} - (6q+3)\binom{2q+2}{k-2q} - (4q+4)\binom{2q+1}{k-(2q+1)} \\
&- (2q+3)\binom{2q}{k-(2q+2)} + (q+1)(6q+2).
\end{aligned}
$$

Proposition 3.7. *Let F_{4_q} be a uniform 4_q-fan with $q \geq 1$. Let δ_k represent the set of all closed sets of size k of F_{4_q} where $|\delta_k| = \alpha - \beta_1 - \beta_2 - \beta_3 + H_k(q)$. Then for $3q+1 \leq k \leq 4q-1$*

$$\begin{aligned} H_k(q) &= (6q^2+8q+2)\binom{q+1}{k-3q} + (6q+4)\binom{q+1}{k-(3q+1)} \\ &+ (6q^2+12q+4)\binom{q}{k-(3q+1)} + (6q+6)\binom{q}{k-(3q+2)}. \end{aligned}$$

Proof. The proof will cover all possible cases of non-closed sets removed more than once. There are five possible ways of removing a non-closed set more than once. We denote $C_{t_i} \cap C_{t_j}$ by G_{ij}. We summarize the choices of edges of the non-closed subsets X of size k in the rows of the tables and each row can be read, without loss of generality, as

	$C_{t_i} - G_{ij}$	$C_{t_j} - G_{ij}$	G_{ij}	$F_{4_q} - (C_{t_i} \cup C_{t_j})$		C_{t_i}	C_{t_j}
X	q	q	q	$k-3q$	$\Rightarrow$	$2q$	$2q$

Let X be a non-closed set with q elements in $C_{t_i} - G_{ij}$, q elements in $C_{t_j} - G_{ij}$, q elements in G_{ij} and $k-3q$ elements in $F_{4_q} - (C_{t_i} \cup C_{t_j})$. Then there are exactly $2q$ elements in C_{t_i} and exactly $2q$ elements in C_{t_j}.

1. We consider the non-closed sets removed more than once with respect to cycles C_{t_1}, C_{t_2} and C_{t_3}. Without loss of generality, we start with the pair $\{C_{t_1}, C_{t_2}\}$.

	$C_{t_1} - G_{12}$	$C_{t_2} - G_{12}$	G_{12}	$F_{4_q} - (C_{t_1} \cup C_{t_2})$		C_{t_1}	C_{t_2}
X	q	q	q	$k-3q$	$\Rightarrow$	$2q$	$2q$
X	$q+1$	$q+1$	$q-1$	$k-(3q+1)$	$\Rightarrow$	$2q$	$2q$

Thus any subgraph of size k of this form is removed twice as a non-closed set with respect to C_{t_1} and with respect to C_{t_2}, since both cycles are of length $2q+1$ but have exactly $2q$ elements in this subgraph. One can easily check that this subgraph has not been removed again with respect to any other cycle. Similarly, we get the same result if we consider a pair of cycles $\{C_{t_2}, C_{t_3}\}$.

Thus the number of repeated non-closed sets of this form is

$$2\left[\binom{q+1}{q}\binom{q+1}{q}\binom{q}{q}\binom{q+1}{k-3q} + \binom{q+1}{q+1}\binom{q+1}{q+1}\binom{q}{q-1}\binom{q+1}{k-(3q+1)}\right].$$

2. We consider the non-closed sets removed more than once with respect to cycle C_{t_i} where $i \in \{1,2,3\}$ and cycle C_{t_j} where $j \in \{4,5\}$.

   (a) Without loss of generality, we start with the pair $\{C_{t_1}, C_{t_4}\}$, where $G_{14} = C_{t_1} \cap C_{t_4}$.

	$C_{t_1} - G_{14}$	$C_{t_4} - G_{14}$	G_{14}	$F_{4_q} - (C_{t_1} \cup C_{t_4})$		C_{t_1}	C_{t_4}
X	$q-1$	q	$q+1$	$k-3q$	$\Rightarrow$	$2q$	$2q+1$
X	q	$q+1$	q	$k-(3q+1)$	$\Rightarrow$	$2q$	$2q+1$

   Thus any subgraph of size k of this form is removed twice as a non-closed set with respect to C_{t_1} and with respect to C_{t_4} since the cycle C_{t_1} has length $2q+1$ but have exactly $2q$ elements in this subgraph and the cycle C_{t_4} has length $2q+2$ but have exactly $2q+1$ elements in this subgraph. One can easily check that this subgraph has not been removed again with respect to any other cycle. Similarly, we get the same result if we consider the pairs of cycles $\{C_{t_2}, C_{t_4}\}, \{C_{t_2}, C_{t_5}\}, \{C_{t_3}, C_{t_5}\}$. Thus the number of repeated non-closed sets of this form is

$$4\left[\binom{q+1}{q+1}\binom{q}{q-1}\binom{q+1}{q}\binom{q+1}{k-3q}+\binom{q+1}{q}\binom{q}{q}\binom{q+1}{q+1}\binom{q+1}{k-(3q+1)}\right].$$

(b) We consider the remaining pairs $\{C_{t_1}, C_{t_5}\}, \{C_{t_3}, C_{t_4}\}$. Without loss of generality, we start with the pair $\{C_{t_1}, C_{t_5}\}$, where $G_{15} = C_{t_1} \cap C_{t_5}$.

	$C_{t_1} - G_{15}$	$C_{t_5} - G_{15}$	G_{15}	$F_{4_q} - (C_{t_1} \cup C_{t_5})$		C_{t_1}	C_{t_5}
X	q	$q+1$	q	$k-(3q+1)$	$\Rightarrow$	$2q$	$2q+1$
X	$q+1$	$q+2$	$q-1$	$k-(3q+2)$	$\Rightarrow$	$2q$	$2q+1$

We get similar results for the pair $\{C_{t_3}, C_{t_4}\}$. Thus the number of repeated non-closed sets of this form is

$$2\left[\binom{q+1}{q}\binom{q+2}{q+1}\binom{q}{q}\binom{q}{k-(3q+1)}+\binom{q+1}{q+1}\binom{q+2}{q+2}\binom{q}{q-1}\binom{q}{k-(3q+2)}\right].$$

3. We consider the non-closed sets removed more than once with respect to cycle C_{t_i} where $i \in \{1, 3\}$ and cycle C_{t_6}. Thus we consider the pairs of cycles $\{C_{t_1}, C_{t_6}\}, \{C_{t_3}, C_{t_6}\}$.

	$C_{t_1} - G_{16}$	$C_{t_6} - G_{16}$	G_{16}	$F_{4_q} - (C_{t_1} \cup C_{t_6})$		C_{t_1}	C_{t_6}
X	q	$q+2$	q	$k-(3q+2)$	$\Rightarrow$	$2q$	$2q+2$
X	$q-1$	$q+1$	$q+1$	$k-(3q+1)$	$\Rightarrow$	$2q$	$2q+2$

Thus using a similar argument from part (1) of the proof, the number of repeated non-closed sets of this form is

$$2\left[\binom{q}{q}\binom{q+2}{q+2}\binom{q+1}{q}\binom{q}{k-(3q+2)}+\binom{q}{q-1}\binom{q+2}{q+1}\binom{q+1}{q+1}\binom{q}{k-(3q+1)}\right].$$

4. We consider the non-closed sets removed more than once with respect to cycle C_{t_j} where $j \in \{4, 5\}$ and cycle C_{t_6}. Thus we consider the pairs of cycles $\{C_{t_4}, C_{t_6}\}, \{C_{t_5}, C_{t_6}\}$.

	$C_{t_4} - G_{46}$	$C_{t_6} - G_{46}$	G_{46}	$F_{4_q} - (C_{t_4} \cup C_{t_6})$		C_{t_4}	C_{t_6}
X	$q-1$	q	$q+2$	$k-(3q+1)$	$\Rightarrow$	$2q+1$	$2q+2$
X	q	$q+1$	$q+1$	$k-(3q+2)$	$\Rightarrow$	$2q+1$	$2q+2$

Thus using similar aurgument from part (1) of the proof, the number of repeated non-closed sets of this form is

$$2\left[\binom{q}{q-1}\binom{q+1}{q}\binom{q+2}{q+2}\binom{q}{k-(3q+1)}+\binom{q}{q}\binom{q+1}{q+1}\binom{q+2}{q+1}\binom{q}{k-(3q+2)}\right].$$

Adding the number of repeated non-closed sets of different forms gives the result. □

Corollary 3.11. *Let F_{4_q} be a uniform 4_q-fan with $q \geq 1$. Let δ_k represent the set of all closed sets of size k of F_{4_q}. Then for $3q+1 \leq k \leq 4q-1$*

$$\begin{aligned}\sum_{k=3q+1}^{4q-1} |\delta_k| &= \sum_{k=3q+1}^{4q-1} \left[\binom{4q+3}{k} - (6q+3)\binom{2q+2}{k-2q} \right] \\ &+ \sum_{k=2q+1}^{3q-1} \left[-(4q+4)\binom{2q+1}{k-(2q+1)} - (2q+3)\binom{2q}{k-(2q+2)} \right] \\ &+ (6q^2+8q+2)\sum_{j=1}^{q-1}\binom{q+1}{j} + (6q+4)\sum_{j=0}^{q-2}\binom{q+1}{j} \\ &+ (6q^2+12q+4)\sum_{j=0}^{q-2}\binom{q}{j} + (6q+6)\sum_{j=0}^{q-3}\binom{q}{j}.\end{aligned}$$

Proposition 3.8. *Let F_{4_q} be a uniform 4_q-fan with $q \geq 1$. Let δ_k represent the set of all closed sets of size k of F_{4_q}. Then for $k = 4q$*

$$|\delta_k| = \frac{1}{3}(2q^3 + 3q^2 + 7q).$$

Proof. It is easy to show that there is no closed set of size $4q$ which does not contain a cycle, therefore it makes sense to count the number of closed sets based on cycles and union of cycles of F_{4_q}. Recall from Notation 3.5, the definition of cycles $C_{t_1}, C_{t_2}, \cdots, C_{t_6}$.

First, we consider the closed sets of size $4q$ with exactly one cycle. We start with closed sets which contain C_{t_1}. Recall C_{t_1} has $2q+1$ edges, thus, the only possibilities of closed sets of size $4q$ is to choose $q-1$ edges in the set $\{a_{31}, a_{32}, \cdots, a_{3q}\}$ of q edges and choose q edges in the set $\{a_{41}, a_{42}, \cdots, a_{4q}, d\}$ of $q+1$ edges. Hence we have $\binom{q}{q-1}\binom{q+1}{q}$ closed sets with respect to cycle C_{t_1}. Similarly, with respect to C_{t_3}.

For cycles $C_{t_2}, C_{t_4}, C_{t_5}, C_{t_6}$, any choice of $2q-1, 2q-2, 2q-2$ and $2q-3$ edges not in the cycle, respectively, added to the respective cycle, will be non-closed. Thus we have $2q(q+1)$ closed sets which contains a single cycle.

Secondly, we consider the closed sets with different unions of cycles $C_{t_1}, C_{t_2}, C_{t_3}, C_{t_4}, C_{t_5}$ of size less than or equal to $4q$. But there are only two distinct unions satisfying the condition, since

$$\begin{gathered}C_{t_1} \cup C_{t_2} = C_{t_1} \cup C_{t_2} \cup C_{t_4} = C_{t_1} \cup C_{t_4} = C_{t_2} \cup C_{t_4}, \\ C_{t_2} \cup C_{t_3} = C_{t_2} \cup C_{t_3} \cup C_{t_5} = C_{t_2} \cup C_{t-5} = C_{t_3} \cup C_{t_5}.\end{gathered}$$

Thus it is enough to show for the closed sets containing $C_{t_1} \cup C_{t_2}$ and the ones containing $C_{t_2} \cup C_{t_3}$. Consider the closed sets containing the union $C_{t_1} \cup C_{t_2}$. Recall that $C_{t_1} \cup C_{t_2}$ has $3q+2$ edges, hence the only possibility of closed sets of size $4q$ is to choose $q-2$ edges in the remaining $q+1$ edges, $\{a_{41}, a_{42}, \cdots, a_{4q}, d\}$. Thus we have $\binom{q+1}{q-2}$ choices. Similarly, if we choose the $3q+2$ edges in $C_{t_3} \cup C_{t_2}$. Hence we have

$$2\binom{q+1}{q-2} = \frac{1}{3}q(q+1)(q-1).$$

Finally, we consider the closed sets which contain the union of C_{t_6} with other cycles of size less than or equal to $4q$. There are two distinct unions satisfying this condition, since

$$C_{t_1} \cup C_{t_6} = C_{t_5} \cup C_{t_6},$$

$$C_{t_3} \cup C_{t_6} = C_{t_4} \cup C_{t_6}.$$

Thus, it is enough to count the closed sets containing $C_{t_1} \cup C_{t_6}$ and the ones containing $C_{t_3} \cup C_{t_6}$. Recall that $C_{t_1} \cup C_{t_6}$ has $3q+3$ edges, hence the only possibility of closed sets of size $4q$ is to choose the remaining $q-3$ edges in $\{a_{21}, a_{22}, \cdots, a_{2q}\}$. Thus we have $\binom{q}{q-3}$ choices. Similarly, if we choose $3q+3$ edges of $C_{t_3} \cup C_{t_6}$. Thus we have

$$2\binom{q}{q-3} = \frac{1}{3}q(q-1)(q-2).$$

Therefore, the total number of closed sets of size $4q$ is

$$2q(q+1) + \frac{1}{3}q(q+1)(q-1) + \frac{1}{3}q(q-1)(q-2) = \frac{1}{3}(2q^3 + 3q^2 + 7q).$$

□

Proposition 3.9. *Let F_{4_q} be a uniform 4_q-fan with $q \geq 1$. Let δ_k represent the set of all closed sets of size k of F_{4_q}. Then for $4q+1 \leq k \leq 4q+3$*

$$|\delta_k| = \begin{cases} 2q^2 & k = 4q+1, \\ 0 & k = 4q+2, \\ 1 & k = 4q+3. \end{cases}$$

Proof. The cases $k = 4q+2$ and $4q+3$ are obvious. Thus, we only consider the case $k = 4q+1$. We follow a similar argument given in the proof of Proposition 3.8. There are no closed sets of size $4q+1$ with exactly one cycle, $2\binom{q+1}{q-1}$ closed sets with unions of smaller cycles, smaller than C_{t_6} and $2\binom{q}{q-3}$ closed sets with unions of other cycles with C_{t_6}. Therefore, the total number of closed sets of size $4q+1$ is

$$2\binom{q+1}{q-1} + 2\binom{q}{q-3} = 2q^2.$$

□

We are now in a position to state the main theorem of this paper.

Theorem 3.12. *Let F_{4_q} be a uniform 4_q-fan with $q \geq 1$. The number of graph compositions of F_{4_q} is given by*

$$C(F_{4_q}) = 2^{4q+3} - (34q+23)2^{2q} + (18q^2 + 46q + 22)2^q - \left[4q^3 + 12q^2 + 16q + 6\right].$$

Proof. By Corollary 1.2

$$\begin{aligned} C(G) &= \sum_{k=0}^{|E(G)|} |\delta_k| \\ &= \sum_{k=0}^{4q-1} |\delta_k| + \sum_{k=4q}^{4q+3} |\delta_k|. \end{aligned} \tag{3.1}$$

Now we treat the sums separately. We start with $\sum_{k=0}^{4q-1} |\delta_k|$, applying Proposition 3.3 and Corollaries {3.6, 3.8, 3.10, 3.11}.

$$\begin{aligned} \sum_{k=0}^{4q-1} |\delta_k| &= \sum_{k=0}^{2q-1} |\delta_k| + |\delta_{2q}| \\ &+ \sum_{k=2q+1}^{3q-1} |\delta_k| + |\delta_{3q}| + \sum_{k=3q+1}^{4q-1} |\delta_k| \\ &= \sum_{k=0}^{2q-1} \binom{4q+3}{k} + \binom{4q+3}{2q} - 3\binom{2q+1}{2q}\binom{2q+2}{0} \\ &+ \sum_{k=2q+1}^{3q-1} \left[\binom{4q+3}{k} - (6q+3)\binom{2q+2}{k-2q}\right] \\ &+ \sum_{k=2q+1}^{3q-1} \left[-(4q+4)\binom{2q+1}{k-(2q+1)} - (2q+3)\binom{2q}{k-(2q+2)}\right] \\ &+ \binom{4q+3}{3q} - (6q+3)\binom{2q+2}{q} \\ &- (4q+4)\binom{2q+1}{q-1)} - (2q+3)\binom{2q}{q-2)} + (q+1)(6q+2) \\ &+ \sum_{k=3q+1}^{4q-1} \left[\binom{4q+3}{k} - (6q+3)\binom{2q+2}{k-2q}\right] \\ &+ \sum_{k=3q+1}^{4q-1} \left[-(4q+4)\binom{2q+1}{k-(2q+1)} - (2q+3)\binom{2q}{k-(2q+2)}\right] \\ &+ (6q^2+8q+2)\sum_{j=1}^{q-1}\binom{q+1}{j} + (6q+4)\sum_{j=0}^{q-2}\binom{q+1}{j} \\ &+ (6q^2+12q+4)\sum_{j=0}^{q-2}\binom{q}{j} + (6q+6)\sum_{j=0}^{q-3}\binom{q}{j}. \end{aligned}$$

Putting like terms together we get

$$
\begin{aligned}
\sum_{k=0}^{4q-1}|\delta_k| &= \sum_{k=0}^{4q-1}\binom{4q+3}{k}-(6q+3)\sum_{i=0}^{2q-1}\binom{2q+2}{i}\\
&- (4q+4)\sum_{i=0}^{2q-2}\binom{2q+1}{i}-(2q+3)\sum_{i=0}^{2q-3}\binom{2q}{i}\\
&+ (6q^2+8q+2)\sum_{j=1}^{q-1}\binom{q+1}{j}+(6q+4)\sum_{j=0}^{q-2}\binom{q+1}{j}\\
&+ (6q^2+12q+4)\sum_{j=0}^{q-2}\binom{q}{j}+(6q+6)\sum_{j=0}^{q-3}\binom{q}{j}+(q+1)(6q+2).\\
&= 2^{4q+3}-\Big[8+21q+24q^2+10q^3+\frac{q}{3}(2q^2+1)\Big]-(6q+3)2^{2q+2}\\
&+ \big[12q^3+36q^2+39q+12\big]-(4q+4)2^{2q+1}+\big[8q^3+20q^2+20q+8\big]\\
&- (2q+3)2^{2q}+\big[4q^3+8q^2+5q+3\big]+(6q^2+8q+2)2^{q+1}\\
&- \big[6q^3+26q^2+26q+6\big]+(6q+4)2^{q+1}-\big[3q^3+11q^2+18q+8\big]\\
&+ (6q^2+12q+4)2^q-\big[6q^3+18q^2+16q+4\big]+(6q+6)2^q\\
&- \big[3q^3+6q^2+9q+6\big]+\big[6q^2+8q+2\big]\\
&= 2^{4q+3}-(6q+3)2^{2q+2}-(4q+4)2^{2q+1}-(2q+3)2^{2q}+(6q^2+8q+2)2^{q+1}\\
&+ (6q+4)2^{q+1}+(6q^2+12q+4)2^q+(6q+6)2^q+\big[6q^2+8q+2\big]\\
&- \Big[8+21q+24q^2+10q^3+\frac{q}{3}(2q^2+1)\Big]+\big[12q^3+36q^2+39q+12\big]\\
&+ \big[8q^3+20q^2+20q+8\big]+\big[4q^3+8q^2+5q+3\big]-\big[6q^3+26q^2+26q+6\big]\\
&- \big[3q^3+11q^2+18q+8\big]-\big[6q^3+18q^2+16q+4\big]-\big[3q^3+6q^2+9q+6\big]\\
&= 2^{4q+3}-(6q+3)2^{2q+2}-(4q+4)2^{2q+1}-(2q+3)2^{2q}+(6q^2+14q+6)2^{q+1}\\
&+ (6q^2+18q+10)2^q-\Big[4q^3+15q^2+18q+7+\frac{q}{3}(2q^2+1)\Big]\\
&= 2^{4q+3}-(34q+23)2^{2q}+(18q^2+46q+22)2^q-\big[4q^3+15q^2+18q+7\big]-\frac{1}{3}(2q^3+q).
\end{aligned}
$$

We now consider the second sum and we apply Proposition 3.8 and Proposition 3.9 to get

$$
\sum_{k=4q}^{4q+3}|\delta_k| = \frac{1}{3}(2q^3+3q^2+7q)+2q^2+1.
$$

Adding $\sum_{k=0}^{4q-1}|\delta_k|+\sum_{k=4q}^{4q+3}|\delta_k|$ we get the result. □

4 Conclusion

In this paper, we have used the principle of inclusion and exclusion to count the number of graph compositions. What other combinatorial techniques can be used in counting graph compositions? Is the number of graph compositions an evaluation of a certain graph polynomial?

Acknowledgement

The first author is fully supported by the National Research Foundation of South Africa, Grant no. 86330.

Competing Interests

The authors declare that no competing interests exist.

References

[1] Knopfmacher A, Mays ME. Graph compositions 1: Basic enumeration. Integers: The Electronic Journal of Combinatorial Number Theory. 2001;1(A04):1-11.

[2] Bajguz W. Graph and union of graph compositions. Adv. Stud. Contemp. Math. (Kyungshang). 2008;16(2):245-249.

[3] Huq A. Compositions of graphs revisited. Electon. J. Combin. 2005;14(1, N15):1-7.

[4] Ridley JN, Mays ME. Compositions of unions of graphs. Fibonacci Quarterly. 2004;42:222-230.

[5] Krzysztof Kwaśniewski A. On compositions of numbers and graphs. Preprint.

[6] Mphako-Banda EG. Graph compositions and flats of cycle matroids. Quaest. Math. 2009;32:523-527.

[7] Tutte TW. Graph theory. Encyclopedia of Mathematics and its Applications, Addison-Wesley Publishing Company, Menlo Park California. 1984;21.

[8] Oxley JG. Matroid theory. Oxford University Press, New York; 1992.

[9] Mphako-Banda EG, Werner S. Graph compositions of suspended Y-trees. (To appear).

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://sciencedomain.org/review-history/13528